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ADMISSIBLE ESTIMATORS, RECURRENT DIFFUSIONS, AND INSOLUBLE BOUNDARY VALUE PROBLEMS[1]

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CONTENTS

1. Introduction.

1.1. *Summary.* Consider the problem of estimating the mean of a multivariate normal distribution on the basis of one observation (or more) from that distribution. Take squared error as the loss function—the mathematically simplest choice, and a frequently studied one. We are interested in determining necessary and sufficient conditions for an estimator, δ, to be admissible.

C. Stein (1956) proved that the best invariant estimator ($\delta(x) = x$) is admissible if m—the dimension of the multivariate normal distribution—satisfies $m \leqq 2$ and is inadmissible if $m \geqq 3$. He also gave a heuristic argument which pleads the case that for sufficiently large m the best invariant estimator must be inadmissible. But this heuristic argument gives no indication of the fact that "sufficiently large" m is really $m = 3$.

There is another interesting division between dimensions $m = 2$ and $m = 3$ with which probabilists and statisticians are familiar. Brownian motion is recurrent in dimensions $m = 1, 2$ and is transient if $m \geqq 3$. A variant of the heuristic argument mentioned above pleads the case that for sufficiently large dimension Brownian motion must be transient, but again there is no indication that $m = 3$ is "sufficiently large."

We have been able to determine a necessary and sufficient condition for an estimator having bounded risk to be admissible. We are also able to extend our considerations to many estimators having unbounded risk.

In the process of establishing this condition we develop a close *mathematical* connection between the statistical question of admissibility and the probabilistic question of recurrence. This connection goes far beyond the invariant cases men-

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tioned above. Roughly, to each "possibly admissible" estimator, δ, there corresponds in a natural way a diffusion on m dimensional space. The indicated result is that the estimator is admissible if and only if the corresponding diffusion is recurrent. As mentioned, we have been able to rigorously establish this result if the estimator has bounded risk, and somewhat more generally. In one direction no regularity conditions are needed: transience of the diffusion implies inadmissibility of the estimator. We remark that for the condition that the estimator has bounded risk there is a natural equivalent condition on the related diffusion. The diffusion related to the best invariant estimator is (essentially) Brownian motion. Therefore the relation between admissibility of the one and recurrence of the other which we described above is a special case of a much more general phenomenon.

The mathematical link between the statistical and the probabilistic problems is a simple calculus of variations minimization problem. The integral involved in the minimization problem is a kind of energy integral. The Euler equation for this minimization problem is an elliptic partial differential equation. This elliptic equation involves the differential generator of the above mentioned diffusion, and it is known that the diffusion is recurrent if and only if the appropriate exterior Dirichlet problem for this equation is insoluble. At the same time, subject to the regularity conditions mentioned above, we are able to exploit the mathematical link to the statistical problem to show that the statistical estimator is admissible also if and only if this exterior Dirichlet problem is insoluble.

The argument leading to our main theorem—Theorem 5.1.1—involves several different steps. For this reason we give a brief outline here of the contents of the paper.

The remainder of Section 1 contains basic definitions used throughout the paper and a sub-section entitled, "A heuristic argument." In this section we describe heuristically the mathematical connection between the statistical and probabilistic problems. At the same time we provide an outline of a possible proof that admissibility of the estimator corresponds to recurrence of the associated diffusion. This outline is partly needed to facilitate the heuristic discussion. It is also hoped that this will aid in an understanding of the proof constructed in later chapters, culminating in Section 5. Some parts of this section are used again later, e.g. in Sub-section 3.1 and in Section 5.

Section 2 contains some material on multivariate Laplace transforms which we have not been able to find elsewhere. These results are needed mainly (but not exclusively) for the multivariate extension of Sacks' theorem (Sacks (1963)) which is proved in Sub-section 3.1. These results may be of some independent interest.

Section 3 contains a variety of preparatory results of a statistical nature. There are several lemmas important for later applications. In addition, Theorem 3.1.1 is the extension of Sacks' theorem mentioned above. Also of interest is Theorem 3.3.1 which provides an alternate characterization of the situation when the estimator has bounded risk, plus a generalization (to the case where the risk is bounded only on a special convex set—K_F).

Section 4 contains a variety of results concerning the diffusions which are associated with decision problems. The considerations of this chapter are mainly probabilistic, although the results are mainly motivated by the related statistical questions. As much as possible this section is written so that it may be read independently of the preceding statistical chapters. We note that Theorem 4.3.1 contains some results which may be of minor probabilistic significance in addition to their statistical usefulness. This theorem provides a test for recurrence of the particular diffusions on E^m which are studied here, as well as some other information. It will be seen that the question of recurrence for the class of diffusions with which we are concerned is relatively much easier than for the general diffusion in several dimensions.

Chapter 5 contains the statement and proof of the main theorem—Theorem 5.1.1. This proof is divided into several sections. The "inadmissibility" half of the theorem is proved in Sub-section 5.2. The proof of the other part of the theorem is concluded in Sub-section 5.7. Theorem 5.6.1 is a minor extension of the Blythe-Stein sufficient condition for admissibility using a method due to R. Farrell.

In Section 6 we describe in more concrete statistical terms the implications of Theorem 5.1.1 by giving some examples of admissible and inadmissible estimators and types of estimators.

1.2. *Basic notation.* Let X be an m-dimensional normal random variable with unknown mean and the identity matrix as variance-covariance matrix. Thus X has density

$$p_\theta(x) = (2\pi)^{-m/2} \exp\left(-\tfrac{1}{2}\textstyle\sum_{i=1}^m (x_i-\theta_i)^2\right)$$

with respect to Lebesgue measure on $E^m = m$ dimensional Euclidean space. Let $\delta = (\delta_1, \cdots, \delta_m)^T$ denote an estimate of $\theta = (\theta_1, \cdots, \theta_m)^T$. We take as loss function

$$L(\theta, \delta) = (\delta-\theta)^T D(\delta-\theta)$$

where D is a fixed, known diagonal $m \times m$ matrix with elements $d_1, d_2, \cdots, d_m$ on the diagonal, $d_1 \geqq d_2 \geqq \cdots \geqq d_m > 0$. Throughout this paper we define the symbol $\| \ \|$ by

$$\|y\|^2 = y^T Dy, \qquad y \in E^m.$$

Note that $\| \ \|$ is the usual norm in E^m only if $D = I$. Thus $L(\theta, \delta) = \|\theta-\delta\|^2$. It will be convenient to have another symbol for the usual norm in E^m. Thus, for $x \in E^m$ define

$$|x|^2 = \textstyle\sum x_i^2.$$

As usual for an estimator $\delta(\cdot)$ the risk function $R(\cdot, \cdot)$ is defined by

$$R(\theta, \delta(\cdot)) = E_\theta L(\theta, \delta(x)).$$

[Note: It can be seen by transforming co-ordinates that the above formulation represents no loss of generality from the situation where X is normal with any known non-singular variance covariance matrix and $L(\cdot, \cdot)$ is any positive definite quadratic form in $(\theta-\delta)$. Also, if there are several independent observations $X^{(1)}, X^{(2)}, \cdots, X^{(n)}$, one of course takes $X = n^{-1} \sum_{i=1}^n X^{(i)}$.]

Let G be any nonnegative Borel measure on E^m. If, in addition, G is a finite measure define the integrated risk of an estimator $\delta(\cdot)$ by

$$B(G, \delta) = \int R(\theta, \delta)G(d\theta).$$

[If $G(E^m) = 1$ $B(G, \delta)$ is of course the Bayes risk of δ with respect to G.] Whether or not G is finite one can define the generalized Bayes estimator δ_G by

$$\delta_G(x) = \frac{\int \theta p_\theta(x)G(d\theta)}{\int p_\theta(x)G(d\theta)} \tag{1.2.1.}$$

so long as the integrals on the right of the above expression exist. (An expression like $\int \theta p_\theta(x)G(d\theta)$, above, is to be interpreted as a vector whose ith co-ordinate is $\int \theta_i p_\theta(x)G(d\theta)$.)

For convenience we define $\gamma_G(x)$ by

$$\gamma_G(x) = \delta_G(x) - x.$$

Define the convolution density $g^* = p^*G$ by

$$g^*(x) = \int p_\theta(x)G(d\theta).$$

Since $\{p_\theta(\cdot)\}$ is an exponential family of distributions the region where $g^*(x) < \infty$ is a convex set. Furthermore on the interior of this region derivatives of g may be computed inside the integral sign in the above expression (Lehmann (1959) page 52). We will be interested only in measures G for which $g^*(x) < \infty$ for all x. For such distributions, differentiating inside the integral sign yields

$$\gamma_G(x) = \frac{\nabla g^*(x)}{g^*(x)} \tag{1.2.2}$$

where, as usual, $(\nabla g^*(x))_i = (\partial/\partial x_i)g^*(x)$.

An estimator δ is called admissible if $R(\theta, \delta') \leqq R(\theta, \delta)$ for all θ implies $R(\theta\ \delta') \equiv R(\theta, \delta)$. [Note: Since L is strictly convex the non-randomized estimators form a complete class among all randomized decision procedures. This justifies our restriction of the above formulation to non-randomized estimators.] It can in fact be shown—see Farrell (1964)—that if δ is admissible and $R(\theta, \delta') \leqq R(\theta, \delta)$ then $\delta = \delta'$ almost everywhere w.r.t. Lebesgue measure. Let us also note here the fact which we prove in Sub-section 3.1 that if δ is admissible then $R(\theta, \delta) < \infty$ for all θ.

If F is a given generalized prior distribution *define* K_F to be the closed convex hull of the support of F. Where the choice of F is clear from the context, as is the case in most parts of this paper, we will write K instead of K_F. In Sub-section 3.5 and following, where F is fixed and known, we shall have occasion to write

$$d(x) = \inf \{|x-y| : y \in K\}, \tag{1.2.3}$$

$$K^\alpha = \{x : d(x) \leqq \alpha\}$$

for $\alpha \geqq 0 (K^0 = K)$. We denote by $\pi(x)$ the unique point of K such that

$$|x-\pi(x)| = d(x), \qquad \pi(x) \in K. \tag{1.2.4}$$

If j: $E^m \to E^1$ we will say j is *piecewise differentiable* if there is a collection of disjoint open sets $0_1, 0_2, \cdots$ such that $\bigcup_{i=1}^{\infty} \bar{0}_i = E^m$ and such that j is continuous on E^m and continuously differentiable at each point in 0_i, $i = 1, 2, \cdots$.

1.3. *A heuristic argument.* In the case of dimension $m = 1$, J. Sacks (1963) has shown that the generalized Bayes procedures form a complete class. (See also R. Farrell (1966).) In Sub-section 3.1 we generalize this result to an arbitrary dimension for the Normal problem at hand. Thus if δ is admissible there is a non-negative measure F such that $f^*(x) < \infty$ for all x and $\delta = \delta_F$. As is now well known, not all procedures of the form δ_F are admissible. See, e.g., Sacks (*op. cit.*). The central aim of this paper is to find necessary and sufficient conditions on the measure F for δ_F to be admissible. Throughout the remainder of this paper F will denote a nonnegative measure with $f^*(x) < \infty$ for all x; and we will be investigating the possible admissibility of δ_F, and related properties.

The fundamental tool for our investigation is the necessary and sufficient condition for admissibility due to C. Stein (1955); see also R. Farrell ((1966) Section 3). According to this, δ_F is admissible only if there is a sequence of non-negative *finite* Borel measures, $G_i, i = 1, 2, \cdots$, satisfying $\mathscr{A}_1 : G_i(\{0\}) = 1$ and $\mathscr{B}_1 : G_i$ has compact support and such that

$$B(G_i, \delta_F) - B(G_i, \delta_{G_i}) \to 0. \tag{1.3.1}$$

Conversely δ_F is admissible if for each $x_0 \in E^m$ there is a sequence G_i satisfying $G_i(\{x_0\}) \geqq 1$ and (1.3.1). In Sub-section 5.6 we show that the following slightly weaker condition also implies admissibility in our problem: If there is a sequence G_i satisfying (1.3.1) and

$$\mathscr{A}_1' : G_i(\{\theta : |\theta| \leqq 1\}) \geqq 1$$

then δ_F is admissible. Note that the condition $\mathscr{B}_1$ is not needed to imply admissibility.

Interchanging the order of integration, and using the definition (1.2.1) we have—as in James and Stein (1960)—for any procedure δ

$$B(G_i, \delta) - B(G_i, \delta_{G_i}) = \int \|\delta(x) - \delta_{G_i}(x)\|^2 g_i^*(x)\, dx \tag{1.3.2}$$

(where $g_i^*(x) = p^*G_i$). Substituting the expression (1.2.2), letting $\delta = \delta_F$, and performing some algebra yields

$$\begin{aligned} B(G_i, \delta_F) &- B(G_i, \delta_{G_i}) \\ &= \int \left\| \frac{\nabla f^*(x)}{f^*(x)} - \frac{\nabla g_i^*(x)}{g_i^*(x)} \right\|^2 g_i^*(x)\, dx \\ &= \int \left\| \frac{f^*(x)\nabla g_i^*(x) - g_i^*(x)\nabla f^*(x)}{(f^*(x))^2} \right\|^2 \frac{(f^*(x))^2}{g_i^*(x)}\, dx. \end{aligned}$$

Defining $\hat{h}_i(x) = g_i^*(x)/f^*(x)$ we have the fundamental equation

$$B(G_i, \delta_F) - B(G_i, \delta_{G_i}) = \int \frac{\|\nabla \hat{h}_i(x)\|^2}{\hat{h}_i(x)} f^*(x)\, dx. \tag{1.3.3}$$

Defining $\hat{\jmath}_i(x) = (\hat{h}_i(x))^{\frac{1}{2}}$ we have the even more useful version

$$B(G_i, \delta_F) - B(G_i, \delta_{G_i}) = \int \|\nabla \hat{\jmath}_i(x)\|^2 f^*(x)\, dx. \tag{1.3.4}$$

Equation (1.3.4) may be viewed as the fundamental equation of this study. The close connection between the statistical problem and diffusions on E^m develops via this equation. We will outline this development below, but first we describe some other aspects of this equation of direct statistical significance.

We begin with some implications of the regularity conditions $\mathscr{A}_1$ and $\mathscr{B}_1$ on the form of $\hat{\jmath}_i$. $\mathscr{A}_1$ implies that $g_i^*(x) \geqq (2\pi)^{-m/2} e^{-\frac{1}{2}}$ when $|x| \leqq 1$. Multiplying F by a positive constant does not affect the value of $R(G_i, \delta_F) - R(G_i, \delta_{G_i})$. Without loss of generality we may thus assume F has been normalized so that $f^*(x) \leqq g_i^*(x)$ for $|x| \leqq 1$. Hence, without loss of generality $\mathscr{A}_1$ implies

$$\mathscr{A}_2 : \hat{\jmath}_i(x) \geqq 1 \qquad \text{for } |x| = 1.$$

If F has compact support then δ_F is an essentially unique Bayes procedure, and hence is admissible. We are therefore only interested in the case where F does not have compact support. In order not to have difficulty here with a situation which needs several special arguments let us assume for the remainder of this section that $K_F = E^m$. In this case a theorem of Birnbaum (1955) proves that $\mathscr{B}_1$ implies

$$\mathscr{B}_2 : \lim_{r\to\infty} \sup_{x:|x|=r} \hat{\jmath}_i(x) = 0.$$

It follows that the existence of a sequence satisfying $\mathscr{A}_2$, $\mathscr{B}_2$, and (1.3.4) is a necessary condition for admissibility. After a few paragraphs—following (1.3.8)—we will give a heuristic argument implying that the existence of such a sequence is also sufficient.

Our procedure to see whether such a sequence $\hat{\jmath}_i$ can possibly exist is to first consider essentially the same problem as above expect that we do not restrict the functions corresponding to $\hat{\jmath}_i$ to be of the special form $(g_i^*(x)/f^*(x))^2$. That is, we consider the problem of minimizing

$$\int_{|x|>1} |\nabla j(x)|^2 f^*(x)\, dx \tag{1.3.5}$$

for piecewise differentiable j subject to the constraints:

$$\mathscr{A}_3 : j(x) \geqq 1 \qquad |x| \leqq 1$$

and

$$\mathscr{B}_3 : \lim_{r\to\infty} \sup_{|x|=r} j(x) = 0.$$

If it is the case that (1.3.5) is bounded below by $c > 0$ for all piecewise differentiable j satisfying $\mathscr{A}_3$ and $\mathscr{B}_3$ then clearly (1.3.4) is bounded below by $cd_m > 0$ when conditions $\mathscr{A}_2$ and $\mathscr{B}_2$ are satisfied. (Note that $\hat{\jmath}$ is certainly piecewise differentiable.)

Hence in this case it follows immediately from the above that δ_F is inadmissible.

The only other case possible is that there is a sequence of functions j_i, each satisfying $\mathscr{A}_3$ and $\mathscr{B}_3$, such that

$$\int |\nabla j_i(x)|^2 f^*(x)\, dx \to 0. \tag{1.3.6}$$

While this indicates that δ_F is admissible, it is very far from proving that fact. In general given functions j_i as above it is usually impossible to find a G_i and associated $\hat{j}_i$ such that $\hat{j}_i = j_i$. It is not even clear at first glance that one can find a $\hat{j}_i$ which approximates the desired j_i. However, under certain conditions this can in fact be done, as we describe in the following paragraph. The line of reasoning of the following paragraph can be made precise with appropriate regularity conditions. We have been more successful in making it precise in dimension $m = 1$ than when $m \geqq 2$. For this reason the argument in Section 5 follows a different and somewhat more involved path. Nevertheless the following heuristic argument is what originally led us to Theorem 5.1.1. Furthermore the proof in Section 5 can be viewed as an attempt to follow the following program with the exception that the approximate equality $\nabla \hat{h}_i(x) \approx \nabla h_i(x)$ for all $x \in E^m$ described below is to be replaced by an equality valid in the mean, rather than everywhere.

When a sequence $\{j_i\}$ satisfying $\mathscr{A}_3$, $\mathscr{B}_3$, and (1.3.6) exists it is reasonable that another sequence $\{j_i'\}$ exists satisfying $\mathscr{A}_3$, $\mathscr{B}_3$, (1.3.6), and

$$\int (j_i'(\theta))^2 F(d\theta) < \infty,$$

and such that the functions j_i are "smooth" in an appropriate sense. We in fact prove such a result in sub-Section 5.4. Define $h_i = (j_i')^2$ and

$$G_i(d\theta) = h_i(\theta)F(d\theta). \tag{1.3.7}$$

There is no loss of generality in assuming that the origin, $0 \in E^m$, has been chosen, and F has been normalized, so that $F(\{\theta : |\theta| \leqq 1\}) \geqq 1$. Thus G_i as defined here is a finite measure satisfying $\mathscr{A}_1'$. Observe that $p_\theta(x)F(d\theta)/f^*(x)$ is a probability distribution with expectation $\delta_F(x)$. Now, make the assumption—which is vital for our argument when $K = E^m$—that $\delta_F(x) - x$ is bounded. Since h_i is a "smooth" function it is therefore reasonable that

$$\hat{h}_i(x) = \int h_i(\theta)p_\theta(x)F(d\theta)/f^*(x)$$

is given approximately by $h_i(\delta_F(x))$, which in turn is approximately $h_i(x)$. While it is harder to verify, it is also reasonable that $\nabla \hat{h}_i(x)$ is approximately $\nabla h_i(x)$. We thus have

$$\begin{aligned} B(\delta_F, G_i) - B(\delta_{G_i}, G_i) &\leqq d_1 \int \frac{|\nabla \hat{h}_i(x)|^2}{h_i(x)} f^*(x)\, dx \\ &\approx d_1 \int \frac{|\nabla h_i(x)|^2}{h_i(x)} f^*(x)\, dx \qquad (1.3.8) \\ &= 4d_1 \int |\nabla j_i(x)|^2 f^*(x)\, dx \to 0 \end{aligned}$$

where "$\approx$" denotes some appropriate sort of approximation. This implies that δ_F is admissible.

In Sub-sections 3.3 and 3.4 we discuss the one basic statistical assumption which is needed for an argument of the above type, namely that $\delta_F(x)-x$ is bounded. In particular we show this is equivalent to the condition that $R(\cdot, \delta_F)$ be bounded.

By the above arguments the question of admissibility of δ_F can therefore be reduced to the question of minimizing

$$\int |\nabla j(x)|^2 f^*(x)\, dx \tag{1.3.5}$$

subject to the boundary conditions $\mathscr{A}_3$ and $\mathscr{B}_3$. This minimization can be discussed from at least two points of view seemingly unrelated to the admissibility of δ_F.

First, one can use the calculus of variations. The Euler equations for the minimization of (1.3.5) can be written

$$f^* \textstyle\sum_{i=1}^m j_{ii}'' + \sum f_i^{*\prime} j_i' = 0 \quad \text{where} \quad j_i' = \dfrac{\partial}{\partial x_i} j, \quad \text{etc.} \tag{1.3.9}$$

It is thus reasonable that (1.3.5) is bounded away from zero if and only if the elliptic partial differential equation—(1.3.9) has a solution for $\{x: |x| > 1\}$ satisfying $\mathscr{A}_3$ and $\mathscr{B}_3$. In Sub-section 4.3 we show that this is the case for $m \geqq 2$. (For $m = 1$, one must treat the left and right halves of the line separately.)

Second, diffusions on E^m are related to equations such as (1.3.5). For purposes of discussion it is perhaps more suggestive to rewrite (1.3.9) as

$$\textstyle\sum j_{ii}'' + \sum \dfrac{f_i^{*\prime}}{f^*} j_i' = 0, \tag{1.3.10}$$

though this is mainly a matter of taste. (Note: If $\delta_F(x)-x$ is bounded then $f_i^{*\prime}(x)/f^*(x)$ is bounded.) Write the left side of (1.3.10) as $\mathscr{L}_F j$ where $\mathscr{L}_F$ is the elliptic partial differential operator. $\mathscr{L}_F$ is the generator of the diffusion (definable on all of E^m) with local variance-covariance matrix $2I$ and local mean $\nabla f^*/f^* = \delta_F(x)-x$. It is easy to check that (1.3.10) has a solution on $\{x: |x| > 1\}$ satisfying $\mathscr{A}_3$ and $\mathscr{B}_3$ if and only if the diffusion is transient. In this case the solution j may be taken to be the probability of ever reaching the unit ball, $\{x: |x| \leqq 1\}$. The indicated result is that (at least when $R(\cdot, \delta_F)$ is bounded) the estimator δ_F is admissible if and only if the related diffusion, defined above, is recurrent. More details of this interpretation are discussed in Section 4.

Note that the usual, best invariant estimator, $\delta_1(x) = x$, is the generalized Bayes estimator for Lebesgue measure as the generalized prior. The diffusion which corresponds to this estimator is a version of Brownian motion. (More precisely, it is exactly the usual Brownian motion run with a $\frac{1}{2}$ speed clock.) We thus have the indicated result that δ_1 is admissible if and only if Brownian motion is recurrent; that is, if and only if $m \leqq 2$. The result that δ_1 is admissible if and only if $m = 1$ or 2 is, of course, already known; see Stein (1956) and (1959), James and Stein (1960), and Brown ((1966) Chapter 3). However we find the connection with Brownian motion interesting and suggestive, if not enlightening.

An interesting question which arises is the following: If $\delta = \delta_F$ is inadmissible, what is a better estimator? In specific cases it is of course possible to find better estimators as in Stein (1960), Brown (1966), and Baranchik (1969). A reasonable conjecture based on the constructions outlined in this section is that if δ_F is inadmissible then the estimator δ_G is better and is admissible where G is the nonnegative measure minimizing $\int |\nabla \tilde{j}(x)|^2 f^*(x)\,dx$ subject to the conditions $G(\{\theta : |\theta| < 1\}) \geqq 1$ and $g^*(x)/f^*(x) \to 0$ as $|x| \to \infty$. Under appropriate conditions such a G exists. By our previous arguments this G will be given approximately (but not exactly) by $j^2(\theta)F(d\theta)$ (if F is normalized by $F(\{\theta : |\theta| < 1\}) = 1$) where j is continuous and satisfies $j(x) \geqq 1$, $|x| \leqq 1$, $j(x) \to 0$ as $|x| \to \infty$, and $\mathscr{L}_F j = 0$ for $|x| > 1$. It may be that under suitable "smoothness" conditions the estimator corresponding to this latter G is also admissible and also improves upon δ_F. We have few explicit results in the above directions, and we do not discuss this conjecture further in this paper, except to point out the following:

If $m \geqq 3$ and $\delta_F(x) = x$ the better estimator suggested by the above considerations is δ_G where G is given *approximately* by

$$G(d\theta) = \begin{cases} d\theta & |\theta| \leqq 1 \\ (1/|\theta|^{2m-4})\,d\theta & |\theta| > 1 \end{cases}$$

(This turns out to be an estimator such that $\delta_G(x) = (1-(2m-4)/|x|^2)x + o(1/|x|)$ as $|x| \to \infty$, which should be compared with those in Stein (1960).) It is interesting to note that if $m \geqq 5$ the measure given above is finite, hence it appears that for $m \geqq 5$ there probably exist proper-Bayes minimax procedures. Recent results of W. Strawderman (1971a, b) appear to confirm some of the above heuristic considerations.

2. Prerequisite results on multivariate Laplace transforms.

2.1. *Convexity of the log of Laplace transforms.* Suppose μ is a finite nonnegative measure on E^m. Define the multivariate Laplace transform $\tilde{\mu}$ by

$$\tilde{\mu}(t) = \int \exp(t \cdot x)\mu(dx) \qquad t \in E^m \tag{2.1.1}$$

when the integral on the right converges and is finite. It is well known that the set of values of t for which $\tilde{\mu}(t)$ exists is convex in E^m. Denote this set by $T(\mu)$. For all t in the interior of $T(\mu)$ the above expression for $\tilde{\mu}(t)$ can be differentiated under the integral sign an arbitrary number of times. Also, $\tilde{\mu}$ is continuous on $T(\mu)$. (See Lehmann (1959) page 52–53).

From these facts we derive:

LEMMA 2.1.1. Log $\tilde{\mu}$ *is a convex function on* $T(\mu)$. *If the support of* μ *is not contained in an* $m-1$ *dimensional subspace of* E^m *then* log $\tilde{\mu}$ *is strictly convex on the interior of* $T(\mu)$.

PROOF. Let $d^2/dt^2(\log \tilde{\mu})$ denote the $m \times m$ matrix whose ijth element is $\partial^2/\partial t_i\, \partial t_j(\log \tilde{\mu})$. Differentiating (2.1.1) twice under the integral sign yields

$$\left[\frac{d^2}{dt^2}(\log \tilde{\mu})\right]_{ij} = \frac{\int x_i x_j \exp(t\cdot x)\mu(dx)}{\int \exp(t\cdot x)\mu(dx)} - \frac{\int x_i \exp(t\cdot x)\mu(dx) \int x_j \exp(t\cdot x)\mu(dx)}{(\int \exp(t\cdot x)\mu(dx))^2} \tag{2.1.2}$$

for t in the interior of $T(\mu)$. Thus $d^2/dt^2(\log \tilde{\mu})$ is the variance co-variance matrix of the random variable with distribution $\exp(t\cdot x)\mu(\cdot)/\int \exp(t\cdot x)\mu(dx)$. Since such a matrix must be positive semi-definite the first sentence of the lemma is proved. If the support of μ is not contained in an $m-1$ dimensional subspace then, from (2.1.2) $(d^2/dt^2)(\log \tilde{\mu})$ is positive definite for all t in the interior of $T(\mu)$. This proves that $\log \tilde{\mu}$ is strictly convex on the interior of $T(\mu)$. The proof of the Lemma is complete.

2.2. *A continuity theorem for Laplace transforms.* The theorem we need is a slightly modified version of the multivariate analog of the usual continuity theorem for (real) Laplace transforms. In the following let P, P', etc. denote closed convex polyhedra in E^m with vertices $(p_1, \cdots, p_k)$, etc. Let int. P denote the interior of P, etc.

In the following we say the sequence $\{q_i\}$ of probability measures on E^m is *uniformly integrable* if $\sup_i \int_{|x|>r} q_i(dx) \to 0$ as $r \to \infty$. When $m = 1$ the probability measures $\{q_i\}$ are uniformly integrable in our sense if and only if they correspond to a family of random variables which is uniformly integrable in the classical sense, see e.g. Doob ((1953) page 629). Various standard theorems concerning uniformly integrable random variables (when $m = 1$) carry over with at most trivial modifications to cover uniformly integrable measures. In particular $\sup_i \int |x|^2 q_i(dx) < \infty$ implies uniform integrability of $\{q_i\}$, and uniform integrability of $\{q_i\}$ implies the existence of a subsequence $\{i_k\} \subset \{i\}$ and a probability measure q_0 such that $q_{i_k} \to q_0$ (weakly).

THEOREM 2.2.1. *Let μ_i be a sequence of finite nonnegative measures. Let P be a closed convex polyhedron such that $P \subset$ int. $T(\mu_i)$ for all i. Suppose $0 \in$ int P. Suppose there is a bound $B < \infty$ such that*

$$\left|\frac{\nabla \tilde{\mu}_i(p_j)}{\tilde{\mu}_i(p_j)}\right| < B \qquad j = 1, 2, \cdots, k;\ i = 1, 2, \cdots. \tag{2.2.1}$$

Let P' be any closed convex polyhedron such that $P' \subset$ int P. Then the probability measures $\exp(t\cdot x)\mu_i(\cdot)/\int \exp(t\cdot x)\mu_i(dx)$ are uniformly integrable over all $t \in P'$ and all $i = 1, 2, \cdots$. Also, there exists a subsequence $\{i'\} \subset \{i\}$ and a probability measure μ such that $\mu_{i'}/\int \mu_{i'}(dx) \to \mu_0$ weakly and $\nabla\tilde{\mu}_{i'}(t)/\tilde{\mu}_{i'}(t) \to \nabla\tilde{\mu}_0(t)/\tilde{\mu}_0(t)$ for all $t \in P'$. Also, $\tilde{\mu}_{i'}(t)/\tilde{\mu}_{i'}(0) \to \tilde{\mu}_0(t)$ for all $t \in P'$.

PROOF. From (2.1.1) it follows that the directional derivative at $t \in$ int $T(\mu_i)$ of $\log \tilde{\mu}_i$ in the direction determined by a unit vector, v, is given by $v\cdot(\nabla\tilde{\mu}_i(t)/\tilde{\mu}_i(t))$.

Since $\log \tilde{\mu}$ is convex this directional derivative must be increasing along a line through v. That is

$$v \cdot \left(\frac{\nabla \tilde{\mu}_i(\alpha v)}{\tilde{\mu}_i(\alpha v)} \right)$$

is a non-decreasing function of $\alpha (\alpha$ real) for $\alpha v \in \operatorname{int} T(\mu_i)$. Let $t_0 \in P$. Substituting $v = (p_j - t_0)/|p_j - t_0|$ in the above we have

$$\left| \frac{p_j - t_0}{|p_j - t_0|} \cdot \left(\frac{\nabla \tilde{\mu}(t_0 + \beta(p_j - t_0))}{\tilde{\mu}(t_0 + \beta(p_j - t_0))} \right) \right| < B \quad \text{for} \quad 0 \leqq \beta \leqq 1. \tag{2.2.2}$$

Integrating (2.2.2) along the line joining t_0 to p_j gives

$$|\log \tilde{\mu}_i(p_j) - \log \tilde{\mu}_i(t_0)| < B|p_j - t_0| \leqq c_1, \qquad j = 1, 2, \cdots, k \tag{2.2.3}$$

where $c_1 = 2 \sup_{1 \leqq j \leqq k} B|p_j - t_0|$. It follows immediately that $\tilde{\mu}_i(p_j)/\tilde{\mu}_i(t_0) \leqq e^{c_1}$.

Let $p_1', \cdots, p_k'$ denote the vertices of P'. Define

$$c_2 = \inf \{|z - P_j'| : z \notin P, j = 1, 2, \cdots, k'\}.$$

Since $P' \subset \operatorname{int} P$, $c_2 > 0$. Using the facts that P' is convex and $|x|$ and $\exp(t \cdot x)$ are convex functions for $t \in P'$ we have

$$\begin{aligned} |x|^2 \exp(t \cdot x) &\leqq |x|^2 \sup_{1 \leqq j \leqq k'} \exp(p_j' \cdot x) \\ &\leqq c_3 (\exp c_2 |x|) \sup_{1 \leqq j \leqq k'} \exp(p_j' \cdot x) \\ &= c_3 \sup_{1 \leqq j \leqq k'} \exp(q_j' \cdot x) \end{aligned}$$

where $q_j' = p_j' + c_2 x/|x|$. The definition of c_2 guarantees that $q_j' \in P$. Hence

$$|x|^2 \exp t \cdot x \leqq c_3 \sup_{1 \leqq j \leqq k} \exp(p_j \cdot x). \tag{2.2.4}$$

Thus, for $t \in P'$,

$$\begin{aligned} \int |x|^2 \exp(t \cdot x) \mu_i(dx) &\leqq c_3 \textstyle\sum_{j=1}^k \int \exp(p_j \cdot x) \mu_i(dx) \\ &= c_3 \textstyle\sum_{j=1}^k \tilde{\mu}_i(p_j), \end{aligned}$$

and

$$\begin{aligned} \int |x|^2 \exp(t \cdot x) \mu_i(dx)/\tilde{\mu}_i(t) &\leqq c_3 \textstyle\sum_{j=1}^k \tilde{\mu}_i(p_j)/\tilde{\mu}_i(t) \\ &\leqq c_3 k\, e^{c_1}. \end{aligned} \tag{2.2.5}$$

It follows that the measures $\exp(t \cdot x) \mu_i(\cdot)/\tilde{\mu}_i(t)$ are uniformly integrable over all $t \in P'$ and $i = 1, 2, \cdots$.

The remaining parts of the theorem follow by standard arguments from this uniform integrability and the fact that

$$(\nabla \tilde{\mu}_i(t))_j = \int x_j \exp(t \cdot x) \mu_i(dx) \qquad j = 1, 2, \cdots, m.$$

This completes the proof of the theorem.

3. Needed statistical results.

3.1. *Generalized Bayes procedures form a complete class.* In this section we give a simple proof of the extension to several dimensions of the theorem of J. Sacks mentioned in Sub-section 1.3. The proof is also valid in dimension $m = 1$. The following explicitly applies only to the problem of Sub-section 1.2, but the method can be generalized to statistical problems involving exponential families of distributions.

THEOREM 3.1.1. *If δ is admissible there is a non negative measure F such that $f^*(x) < \infty$ for all x and such that $\delta(x) = \delta_F(x)$* a.e. (dx).

PROOF. If G_i is a sequence satisfying $\mathscr{A}_1$ then $g_i^*(x) \geqq (2\pi)^{-m/2} \exp(-|x|^2/2)$. Hence (1.2.2) and (1.3.2) imply $\nabla g_{i'}^*(x)/g_{i'}^*(x) \to \gamma(x)$ in measure (dx) on each compact set in E^m. Thus there is a subsequence i' such that

$$\|\nabla g_{i'}^*(x)/g_{i'}^*(x)\| \to \|\gamma(x)\| < \infty \text{ a.e. } (dx).$$

Defining (for this sub-section only) $\mu_i(d\theta) = \exp(-|\theta|^2/2)G_i(d\theta)$, the above implies $\nabla\tilde{\mu}_{i'}(x)/\tilde{\mu}_{i'}(x) \to x+\gamma(x) < \infty$ a.e. (dx). Thus, using Theorem 2.2.1 there is a measure $\mu_0(d\theta)$ and a subsequence $\{i''\}$ of $\{i'\}$ such that $\nabla\tilde{\mu}_{i''}(x)/\tilde{\mu}_{i''}(x) \to \nabla\tilde{\mu}_0(x)/\tilde{\mu}_0(x)$ for all $x \in E^m$. It follows that $\nabla\tilde{\mu}_0(x)/\tilde{\mu}_0(x) = x+\gamma(x)$ a.e. (dx). Defining $F(d\theta) = \exp(|\theta|^2/2)\mu_0(d\theta)$ we compute $f^*(x) < \infty$ and $\nabla f^*(x)/f^*(x) = \nabla\tilde{\mu}_0(x)/\tilde{\mu}_0(x) - x$ and hence $\delta(x) = \delta_F(x)$ a.e. (dx). This completes the proof of the theorem.

The construction in the proof of Theorem 3.1.1 and Lemma 2.1.1 yield the following important Lemma.

LEMMA 3.1.2. *If $\alpha > 0$*

$$\delta_F(x+\alpha y)\cdot y \geqq \delta_F(x)\cdot y. \tag{3.1.1}$$

[In words: the y co-ordinate of δ_F is non-decreasing as one travels in the y-direction.]

PROOF. Defining $\mu_0(d\theta) = \exp(-|\theta|^2/2)F(d\theta)$ as in the previous proof we have

$$\delta_F(x) = \nabla(\log\tilde{\mu}_0(x)).$$

(3.1.1) then follows immediately from Lemma 2.1.1.

From this lemma we obtain, among other results,

LEMMA 3.1.3. *If $|\gamma_F(x)| < B$ for all $x \in K$, then for $x \notin K$. $\gamma_F(x)\cdot(\pi(x)-x)/d(x) \leqq B+d(x)$. Hence, for $x \notin K$*

$$f^*(x) \geqq \exp(-Bd(x)-d^2(x)/2)f^*(\pi(x)). \tag{3.1.2}$$

PROOF. The first statement of the lemma is an immediate corollary of Lemma 3.1.2. Simply take $y = \pi(x)-x$ in (3.1.1) and observe that for $x \in K$, $\gamma_F(x)\cdot z/|z| \leqq B$ for any z. The inequality (3.1.2) results from integrating the first inequality of the lemma along the line from $\pi(x)$ to x. The lemma is proved.

Before closing this section we also note the following relation

LEMMA 3.1.4.

$$f^*(x) \leqq \exp(-d^2(x)/2) f^*(\pi(x)). \tag{3.1.3}$$

PROOF. For $\theta \in K$

$$|x-\theta|^2 \geqq d^2(x) + |\pi(x)-\theta|^2.$$

Hence

$$\begin{aligned} f^*(x) &= (2\pi)^{-m/2} \int \exp(-|x-\theta|^2/2) F(d\theta) \\ &\leqq (2\pi)^{-m/2} \exp(-d^2(x)/2) \int \exp(-|\pi(x)-\theta|^2/2) F(d\theta) \\ &= \exp(-d^2(x)/2) f^*(\pi(x)). \end{aligned}$$

3.2. *Lemmas.* The following condition will play a key role in Section 5 where it is numbered as condition (5.1.1). There exists a $B < \infty$ such that

$$|\gamma_F(x)| < B \qquad \text{for all } x \in K. \tag{3.2.1}$$

In the next section we give an alternative characterization of the situation when this condition is satisfied. In this section we prove some lemmas to prepare for this characterization. The second of these lemmas and its corollary also play an independent role in the proof in Section 5.

The first lemma expresses an important although nearly trivial fact. We remind the reader that K is the closed convex hull of the support of F.

LEMMA 3.2.1. *For all x, $\delta_F(x) \in K$.*

REMARK. The converse of this result is also true. To be precise:

$$K = \overline{\{\delta_F(x) : x \in E^m\}}.$$

This latter result is not quite as trivial as Lemma 3.2.1 and we do not need it for our development. Hence we do not give its proof here.

PROOF OF LEMMA 3.2.1.

$$\delta_F(x) = \int \theta p_\theta(x) F(d\theta)/f^*(x).$$

$p_\theta(x)F(d\theta)/f^*(x)$ is the mass element of a probability distribution whose support is contained in the convex set K. Hence $\delta_F(x) \in K$. This completes the proof.

LEMMA 3.2.2. *Suppose* (3.2.1) *is satisfied. Then given $k < \infty$ there exists a constant $\zeta < \infty$ (depending only on k, B, and m) such that*

$$\int \exp(k|\theta - x|) p_\theta(x) F(d\theta)/f^*(x) \leqq \zeta \exp(\zeta d(x)) \tag{3.2.2}$$

for all $x \in E_m$.

PROOF. To begin, note that

$$\exp(k|\theta_i - x_i|) p_\theta(x) \leqq \exp(k^2/2) \textstyle\sum_{j=1}^{2} p_\theta(x + (-1)^j k e_i) \tag{3.2.3}$$

where e_i is the unit vector in the ith co-ordinate direction. Then, since $|\theta - x| \leqq \sum|\theta_i - x_i|$

$$\int \exp(k|\theta - x|)p_\theta(x)F(d\theta)/f^*(x) \leqq \exp(k^2/2)\sum_{j=1}^{2}\sum_{i=1}^{m} f^*(x+(-1)^j ke_i)/f^*(x). \tag{3.2.4}$$

If $x \in K$ and $|y-x| \leqq k$ then $|\pi(y)-x| \leqq k$. Hence using (3.2.1) and Lemma 3.1.4 (or Lemma 3.1.2), for all $x, y \in E^m$, $|y-x| \leqq k$ implies

$$f^*(y)/f^*(x) \leqq f^*(\pi(y))/f^*(x) \leqq e^{Bk}.$$

It follows from this and (3.2.4) that if $x \in K$

$$\int \exp(k|\theta - x|)p_\theta(x)F(d\theta)/f^*(x) \leqq 2m\exp(k^2/2 + Bk). \tag{3.2.5}$$

If $x \notin K$ we proceed as follows. By rotation and translation of co-ordinates we may assume without loss of generality that $x = (x_1, 0, \cdots, 0)$ with $-d(x) = x_1 < 0$, and $\pi(x) = 0$, and $K \subset \{\theta = (\theta_1, \cdots, \theta_m) : \theta_1 \geqq 0\}$. For $\theta \in K$ $\theta \cdot x \leqq 0$ and $|x-\theta| \leqq d(x) + |\theta|$. Hence

$$\begin{aligned} &\int \exp(k|x-\theta|)p_\theta(x)F(d\theta) \\ &\leqq (2\pi)^{-m/2}\int \exp(k\,d(x) + k|\theta| - |\theta - x|^2/2) \\ &\leqq (2\pi)^{-m/2}\int \exp(k\,d(x) - |x|^2/2)\exp(k|\theta| - |\theta|^2/2)F(d\theta) \\ &= \exp(k\,d(x) - d^2(x)/2)2m\exp(k^2/2 + Bk)f^*(0) \end{aligned} \tag{3.2.6/7}$$

where for the last step we have used (3.2.5). From Lemma 3.1.3

$$f^*(x) \geqq f^*(0)\exp(-d^2(x)/2 - Bd(x)). \tag{3.2.8}$$

Hence for $x \notin K$

$$\int \exp(k|x-\theta|)p_\theta(x)F(d\theta)/f^*(x) \leqq \exp((k+B)d(x))2m\exp(k^2/2+Bk). \tag{3.2.9}$$

Letting $\zeta = \max(2m\exp((k+B)^2/2), k+B)$ the condition (3.2.2) is satisfied. This completes the proof of the lemma.

LEMMA 3.2.3. *Suppose* (3.2.1) *is satisfied. Then there is a constant* ζ_1 *(depending only on* B *and* m*) such that*

$$|\gamma_F(x)| \leqq \zeta_1(1 + d(x)). \tag{3.2.10}$$

PROOF. $\gamma_F(x)$ is the expectation of $\theta - x$ under the distribution described by $p_\theta(x)F(d\theta)/f^*(x)$. Hence by Jensen's inequality

$$\exp|\gamma_F(x)| \leqq \int \exp(|\theta - x|)p_\theta(x)F(d\theta)/f^*(x).$$

Thus by Lemma 3.2.2, there is a $\zeta < \infty$ such that

$$\exp|\gamma_F(x)| \leqq \zeta\exp(\zeta d(x)).$$

It follows that

$$|\gamma_F(x)| \leqq \log \zeta + \zeta d(x).$$

Setting $\zeta_1 = \zeta$ completes the proof of the corollary.

For the next lemma we do not assume that (3.2.1) is satisfied. In fact the use of this lemma is mainly in situations where (3.2.1) is not satisfied.

LEMMA 3.2.4. *There is a constant ζ_3 (depending only on m and d_m) such that*

$$R(x, \delta_F) \geqq \zeta_3(|\gamma_F(x)|-2)^2 \qquad \text{for } \gamma_F(x) \geqq 2. \tag{3.2.11}$$

Note: The significant part of the above lemma is not the exact form of (3.2.11) (which is not the best form possible) but rather the fact that $|\gamma_F(x)|$ large implies that $R(x, \delta_F)$ is also large.

PROOF. By translating and rotating co-ordinates we may assume that $x = (0, 0, \cdots, 0)$ and $\gamma_F(x) = (\gamma_1, 0, \cdots, 0)$ where $\gamma_1 = |\gamma_F(x)|$. Consider any unit vector ρ, say, from the origin and making an angle less than 45° with the positive x axis. $\rho \cdot \gamma_F(x) \geqq \gamma_1/2^{\frac{1}{2}}$. For any point $z = k\rho$, $0 < k < 1$. Lemma 3.1.2 implies that $\rho \cdot (z+\gamma(z)) \geqq \gamma_1/2^{\frac{1}{2}}-1$. Let Q denote here the set of all such points z.

$$\begin{aligned} R(x, \delta_F) &= \int \|y+\gamma_F(y)\|^2 p_0(y)\, dy \\ &\geqq \int_Q \|y+\gamma_F(y)\|^2 \inf_{|y| \leqq 1} p_0(y)\, dy \\ &\geqq d_m(2\pi)^{-m/2} e^{-1} [(\gamma_1/2^{\frac{1}{2}}-1)^2] \int_Q dy \\ &\geqq \zeta_3(|\gamma(x)|-2)^2. \end{aligned}$$

This completes the proof of the lemma.

3.3. *A condition equivalent to bounded risk.*

THEOREM 3.3.1. *There is a constant R such that*

$$R(\theta, \delta_F) < R \qquad \text{for all } \theta \in K \tag{3.3.1}$$

if and only if there is a constant B such that $|\gamma_F(x)| < B$ for all $x \in K$.

PROOF. Suppose (3.2.1) is satisfied. Note that for $\theta \in K$, $|\theta - x| \geqq d(x)$. Using Lemma 3.2.3, for $\theta \in K$

$$\begin{aligned} R(\theta, \delta_F) &= \int \|x+\gamma_F(x)-\theta\|^2 p_\theta(x)\, dx \\ &\leqq 2d_1 \int (|x-\theta|^2+|\gamma_F(x)|^2) p_\theta(x)\, dx \\ &\leqq 2d_1 m + 2d_1 \zeta_1{}^2 \int (1+|\theta-x|)^2 p_\theta(x)\, dx \\ &\leqq d_1(2m+4\zeta_1{}^2+4m\zeta_1{}^2) \end{aligned}$$

where ζ_1 is as in Lemma 3.2.3.

Conversely, suppose (3.2.1) is not satisfied. Then for any $b > 0$ there is an $x \in K$ such that $|\gamma_F(x)| > b$. It follows from Lemma 3.2.4 that $\sup_{\theta \in K} R(\theta, \delta_F) = \infty$, and hence (3.3.1) is not satisfied.

This completes the proof of the theorem.

The following corollary is also of interest.

COROLLARY 3.3.2. *$R(\cdot, \delta_F)$ is bounded (i.e. $\sup_{\theta \in E^m} R(\theta, \delta_F) < \infty$) if and only if $\sup_{x \in E^m} |\gamma_F(x)| < \infty$. ($\sup_{x \in E^m} |\gamma_F(x)| < \infty$ only if $K = E^m$.)*

PROOF. It follows without difficulty from Lemma 3.2.1 that if $K \neq E^m$ then $\sup_{\theta \in E^m} R(\theta, \delta_F) = \infty$. After this observation, the corollary is a direct application of Theorem 3.3.1.

3.4. *Inequalities for the case where F is absolutely continuous.* It is not always easy to determine from a knowledge of F if the condition (3.2.1) is satisfied. In this section we give a sufficient condition which applies when $K = E^m$ and F has a sufficiently smooth density. Similar (but not identical) results are also valid when $K \neq E^m$ but we do not give such results here. Lemma 3.4.1, below, is also used in Sub-section 5.3 in the proof of the main theorem.

Throughout this section we assume F has a density with respect to Lebesgue measure, which we denote by f.

LEMMA 3.4.1. *Suppose f is a continuously differentiable function satisfying*

$$|\nabla f(x)|/f(x) \leqq c \qquad \text{for all } x \in E^m \tag{3.4.1}$$

for some $c < \infty$. Then

$$(e^{c^2}+2^{m/2})f(x) \geqq f^*(x) \geqq (2\pi)^{-m/2}\, e^{-c-\frac{1}{2}} f(x) \tag{3.4.2}$$

and

$$|\nabla f^*(x)|/f^*(x) \leqq c. \tag{3.4.3}$$

PROOF. If $|y-x| \leqq 1$ then (3.4.1) implies $f(y) \geqq e^{-c}f(x)$. Hence

$$\begin{aligned} f^*(x) &> \textstyle\int_{|y-x|<1} f(y)p_x(y)\, dy \\ &> e^{-c}f(x) \textstyle\int_{|z|<1} p(z)\, dz \geqq (2\pi)^{-m/2}\, e^{-c-\frac{1}{2}} f(x), \end{aligned}$$

which verifies the right-hand inequality in (3.4.2).

Similarly, $f(y) \leqq \exp(c|y-x|) f(x)$. Hence

$$f^*(x) \leqq f(x) \textstyle\int \exp(c|\theta-x|)p_\theta(x)\, d\theta.$$

Utilizing the fact that $c|\theta-x| \leqq c^2 + |\theta-x|^2/4$ we compute that

$$\textstyle\int \exp(c|\theta-x|)p_\theta(x)\, d\theta \leqq e^{c^2}+2^{m/2}.$$

Substituting this in the above yields the left-hand inequality in (3.4.2).

In order to prove (3.4.3) we begin by assuming without loss of generality that $x = 0$ and $\nabla f^*(x) = (|\nabla f^*(x)|, 0, \cdots, 0)$. For this proof let

$$r(t) = \int f((t, x_2, \cdots, x_m))p(x_2, \cdots, x_m)(0) \prod_{i=2}^{m} dx_2.$$

From (3.4.1) it follows that r is continuously differentiable and

$$|r'(t)|/r(t) \leqq c. \tag{3.4.4}$$

Now,

$$f^*(x) = (1/2\pi)^{\frac{1}{2}} \int r(t)\, e^{-t^2/2}\, dt \tag{3.4.5}$$

and

$$|\nabla f^*(x)| = |(1/2\pi)^{\frac{1}{2}} \int \mathrm{tr}(t)\, e^{-t^2/2}\, dt|. \tag{3.4.6}$$

From (3.4.4) to (3.4.6) the maximum value of $|\nabla f^*(x)|/f^*(x)$ occurs if $r(t)$ is of the form Ce^{ct} (use the calculus of variations), from which we easily compute that (3.4.3) is satisfied. This completes the proof.

Note that generalizations of Lemma 3.4.1 are used in Section 6 without further proof.

3.5. *The boundary condition at* ∞. In Sub-section 1.3 we described a boundary condition $\mathscr{B}_2$, at ∞ satisfied by $\hat{j}(x)$ when $K = E^m$ and $\mathscr{B}_1$ is satisfied. In this section we prove this result as well as an appropriate extension if $K \neq E^m$.

The results of this section are vacuous if K is compact, so we may as well assume throughout this section that K is not compact. Then, since K is convex, for all sufficiently large r there is an $x \in K$ such that $|x| = r$.

The following two paragraphs are used only in this section and in Sub-section 4.2.

Assuming K is not compact, if $x_0 \in K$ there is another point x_1, say, such that $x_1 \in K$, $|x_1 - x_0| = 1$, and the ray

$$P_{x_0,x_1} = \{y : y = x_0 + \alpha(x_1 - x_0),\ \alpha \geqq 0\}$$

satisfies $P_{x_0,x_1} \subset K$. For a given $x_0 \in K$ let $Q_1^{x_0}$ be the set of all such points x_1; i.e.,

$$Q_1^{x_0} = \{x_1 : |x_1 - x_0| = 1,\ \{y : y = x_0 + \alpha(x_1 - x_0),\ \alpha \geqq 0\} \subset K\}.$$

Since K is closed and convex, $Q_1^{x_0}$ is compact (for more reasoning see (3.5.1)). Let

$$Q_2^{x_0} = \{y : |y - x_0| = 1,\ \exists x \in Q_1^{x_0} \ni |y - x| \leqq 1\}.$$

Clearly $Q_2^{x_0}$ is also compact. As above, for any two points y, z define the ray $P_{y,z}$ by

$$P_{y,z} = \{w : w = y + \alpha(z - y),\ \alpha \geqq 0\}.$$

Let

$$Q_3^{x_0} = \{z : \exists\, y \in Q_2^{x_0} \ni z \in P_{x_0,y}\}.$$

We remind the reader that for $\alpha \geqq 0$

$$K^\alpha = \{x : d(x) \leqq \alpha\} \qquad (K^0 = K).$$

Let

$$L^{x_0}_{\alpha,r} = \{x : |x - x_0| = 1, x_0 + r(x - x_0) \in K^\alpha\}.$$

For fixed α, $r < s$ implies $L^{x_0}_{\alpha,r} \supset L^{x_0}_{\alpha,s}$. Also, from the geometry of the situation

$$\bigcap_{r>0} L^{x_0}_{\alpha,r} = Q_1^{x_0} \subset Q_2^{x_0}. \tag{3.5.1}$$

It follows from (3.5.1) that given x_0 and $\alpha \geqq 0$ there is an R_{α,x_0} sufficiently large so that

$$|x - x_0| > R_{\alpha,x_0} \quad \text{and} \quad x \in K^\alpha \Rightarrow x \in Q_3^{x_0}. \tag{3.5.2}$$

LEMMA 3.5.1. *If G has compact support and α is any fixed number, $\alpha \geqq 0$, then*

$$\lim_{r\to\infty} \sup_{\{x:\ x \in K^\alpha, |x| \geqq r\}} (g^*(x)/f^*(x)) = 0. \tag{3.5.3}$$

Note: The supremum in (3.5.3) may be taken over a larger set than indicated in (3.5.3) (for example over $\{x : x \in K^\alpha \cup Q_3^{x_0}, |x| \geqq r\}$.) However (3.5.3) is the form we will use later in Sub-sections 4.2 and 5.2, and we prove only (3.5.3) in the following.

PROOF. Let us first suppose that $x_0 = 0$. For this proof let $\{(r, \theta)\}$ be the spherical co-ordinate system with $r = |x| \geqq 0$, θ a point on the unit sphere in E^m, and $(r, \theta) = r\theta$.

Let ρ_1 be any ray of the form $\{(r, \theta_1) : r \geqq 0, \theta_1 \text{ fixed}\}$ such that $\rho_1 \subset K$ (i.e. $\theta_1 \in Q_1^{\ 0}$). The fact that K is closed convex and non-compact and $0 \in K$ guarantees as noted above that at least one such ray exists. Let θ_2 be any point on the unit sphere such that $|\theta_2 - \theta_1| \leqq 1$ (i.e. $\theta_2 \in Q_2^{\ 0}$) and let $\rho_2 = \{(r, \theta_2) : r \geqq 0\}$. Thus the angle between ρ_1 and ρ_2 is between $-\pi/3$ and $+\pi/3$ inclusive.

Consider $\lim_{r\to\infty} g^*(r\theta_2)/f^*(r\theta_2)$. There is a hyperplane—call it H_2 (for this paragraph, only) such that H_2 is orthogonal to ρ_2, and supp G is on one side—call it S_1—of H_2, and for all r sufficiently large the points $r\theta_2$ are on the other side—call it S_2—of H_2. S_i, $i = 1, 2$, are disjoint open half spaces. Since $\rho_1 \subset K$ it follows that supp $F \cap S_2 \neq \varphi$. Hence it follows from a result of Birnbaum (1955) that

$$\lim_{r\to\infty} g^*(r\theta_2)/f^*(r\theta_2) = 0. \tag{3.5.4}$$

Thus (3.5.4) holds for each $\theta_2 \in Q_2^{\ 0}$. For each r, $g^*(r\theta)/f^*(r\theta)$ is a continuous function on the compact set $Q_2^{\ 0}$. Hence

$$\lim_{r\to\infty} \sup_{\theta \in Q_2^0} g^*(r\theta)/f^*(r\theta) = 0.$$

Equivalently

$$\lim_{r\to\infty} \sup_{\{x\,:\,|x|=r,\, x \in Q_3^0\}} g^*(x)/f^*(x) = 0. \tag{3.5.5}$$

But, from (3.5.2) for all r sufficiently large $\{x : |x| = r, x \in Q_3^{\ 0}\} \supset \{x : |x| = r, x \in K^\alpha\}$. Hence (3.5.5) implies (3.5.3) is satisfied.

If $x_0 \neq 0$ translate the co-ordinate system so that the origin of the new system is x_0. Let $|\cdot|'$ denote the norm in the new system. From (3.5.5) we have

(3.5.6) $$\lim_{r\to\infty} \sup_{\{x\,:\,|x|'\geqq r,\ x\in K^\alpha\}} g^*(x)/f^*(x) = 0.$$

But for r sufficiently large $|x|' \geqq r$ implies $|x| \geqq r/2$. Hence (3.5.3) follows immediately from (3.5.6). This completes the proof.

4. The diffusion $\{Z_t\}$ and the minimization problem.

4.1. *Statement of the problem, definitions, preliminary remarks.* Let F, f^* be as in Sub-section 1.2 and $K = K_F$. We define the following boundary conditions:

(4.1.1) $$j(x) \geqq 1 \qquad |x| \leqq 1$$

and

(4.1.2) $$\lim_{r\to\infty} \sup_{\{x\,:\,x\in K^m,\ |x|=r\}} j(x) = 0.$$

[Note that (4.1.2) is vacuous if K is compact.] [The reason for using $K^\beta = \{x : d(x) \leqq \beta\}$ rather than K in (4.1.2) will become clear at Lemma 4.2.2. We could substitute any $\beta > 0$ for m in (4.1.2) but the choice m is convenient in Lemma 4.2.1. See the note following (4.2.4).] Let J denote the set of all piecewise differentiable functions satisfying (4.1.1) and (4.1.2). We consider in this Section the problem of finding

(4.1.3) $$\inf_{j\in J} \int |\nabla j(x)|^2 f^*(x)\, dx.$$

In particular, we are mainly interested in determining whether this infimum is 0 or is greater than 0.

As one means of describing when the infimum in (4.1.3) is 0 we introduce—as in Sub-section 1.3—the diffusion $\{Z_t\}$ with local mean $\nabla f^*(x)/f^*(x)$ and local variance $2I$. This diffusion is also a useful tool for studying certain aspects of the minimization problem. In general our results will not depend strongly on the starting point of $\{Z_t\}$, but where we wish to indicate that the diffusion starts at a point x at time $t = 0$ we will write $\{Z_t^x\}$ to indicate that fact. (In Section 5 the symbol $\{Z_t^1\}$ will have a different meaning.) We will use without further comment the well-known fact that $\{Z_t\}$ has the strong Markov property. Strictly speaking, we should write $\{Z_t(\omega)\}$ instead of $\{Z_t\}$ in all the above arguments, where $\omega \in \Omega$ and Ω is a suitable probability space. For simplicity of notation we have omitted the symbol—(ω)—from Z_t and all other random variables to be defined later in this chapter.

We say that $\{Z_t\}$ is *recurrent* if for *all* $x \in E^m$ the function $\mathscr{H}$ defined by

(4.1.4) $$\mathscr{H}(x) = \Pr\{\inf_t |Z_t^x| \leqq 1\}$$

satisfies $\mathscr{H}(x) = 1$. In words, $\{Z_t\}$ is *recurrent* if for all $x \in E^m$ the diffusion $\{Z_t^x\}$ hits the unit sphere with probability one. If $\{Z_t\}$ is not recurrent it is *transient*.

It follows by standard arguments from the definition of $\{Z_t\}$ that if $m \geqq 2$ and $\{Z_t\}$ is transient then for all x with $|x| > 1$

$$\mathscr{K}(x) = \Pr\{\inf_t |Z_t^x| \leqq 1\} < 1. \tag{4.1.5}$$

If $m = 1$ and $\{Z_t\}$ is transient either $\mathscr{K}(x) = \Pr\{\inf_t |Z_t^x| \leqq 1\} < 1$ for all $x > 1$ or for all $x < -1$ (or both).

Before proceeding further we note that some of the results of this chapter generalize to the case where f^* is replaced by a "smooth" positive function, say f, but we do not concern ourselves here with such possible generalizations.

In order to answer the question posed at (4.1.3) we will also have occasion to consider the related problem of minimizing

$$\int_{x \in O} |\nabla k(x)|^2 f^*(x)\, dx \tag{4.1.6}$$

for piecewise differentiable functions satisfying

$$k(x) = k'(x) \qquad \text{for } x \notin O \tag{4.1.7}$$

where O is a bounded open set and k' is a given nonnegative piecewise differentiable function. If the infimum is finite a unique minimizing solution always exists for this problem. It is the unique function satisfying (4.1.7) and

$$\sum_{i=1}^{m} k_{ii}''(x) + \sum_{i=1}^{m} \left(\frac{f_i^{*\prime}(x)}{f^*(x)}\right) k_i'(x) = 0 \tag{4.1.8}$$

for $x \in O$ where $(k_i'(x) = (\partial/\partial x_i)k$, etc.). Note, $k \geqq 0$.

We will have occasion to use the following version of Harnack's inequality (see, e.g., Serrin (1956)). If k satisfies (4.1.8) for all $x \in O$ and $k \not\equiv 0$ and $C \subset O$ is compact then there is a $b < \infty$ such that

$$\sup_{x \in C} (|\nabla k(x)|/k(x)) < b < \infty. \tag{4.1.9}$$

We note that b in (4.1.9) need not be a function of O or k' if O is sufficiently large. More precisely, let $O' \supset C$ be an open set. Then the bound b may be chosen so that (4.1.9) is satisfied for all $O \supset O'$ and all nonnegative k'.

4.2. *Lemmas.* In this section we prove some lemmas which we need for the proof of Theorem 4.3.1. Lemma 4.2.1 is strictly probabilistic, and provides the key to our proof of Lemma 4.2.2, whose conclusion is of an analytic nature. Lemma 4.2.3 applies only in the case where $\{Z_t\}$ is recurrent. Its conclusion is similar in nature to that of Lemma 4.2.2, but is somewhat simpler.

For this section and Sub-section 4.3 we will use the following definitions: $O^R = \{x : |x| > 1 \text{ and } |x| < R \text{ or } x \notin K^m\}$. ($O^R$ is open, but not necessarily bounded.)

$$T_R^x = \inf\{t : Z_t^x \notin O^R\}.$$

Note that T_R^x is a random quantity since it is a function of the sample path.

LEMMA 4.2.1. *Fix $R < \infty$. For any $x \in O^R$*

$$\Pr\{T_R^x < \infty\} = 1. \tag{4.2.1}$$

PROOF. We begin by noting that it can be computed from the infinitesimal mean and variance of Z_t that if $x \notin K$

$$(4.2.2) \qquad \lim_{t\to 0} E(d(Z_t^x)-d(x))/t = -\nu(x)+\frac{m-1}{d(x)} \leqq -d(x)+\frac{m-1}{d(x)}$$

where

$$\nu(x) = |(x-\pi(x))\cdot\gamma(x)|/d(x) \geqq d(x).$$

If $x \notin K^m$ we have from (4.2.2)

$$(4.2.3) \qquad \lim_{t\to 0} E(d(Z_t^x)-d(x))/t \leqq -1.$$

For this paragraph only let

$$\tau^x(t) = \inf(t, \inf\{t: Z_t^x \in K^m\}).$$

It follows from (4.2.3) that for $x \notin K^m$ $d(Z^x_{\tau^x(t)})$ is a continuous super-martingale bounded below by m. Furthermore, for any x

$$(4.2.4) \qquad \Pr\{\exists t \ni Z_t^x \in K^m\} > 0.$$

[*Note*: (4.2.4) remains true if the superscript m on K^m is replaced by any $\beta > 0$ though the above reasoning does not quite supply a proof. However, it may be false if m is replaced by 0. For example, suppose $m = 3$ and $K = K^0$ is a one-dimensional subspace; then $\Pr\{\exists t \ni Z_t^x \in K^0\} = 0$ if $x \notin K$.] (4.2.4) and the super-martingale property imply that for all x

$$(4.2.5) \qquad \Pr\{\exists t \ni Z_t^x \in K^m\} = 1,$$

see e.g. Lamperti (1960, Theorem 2.1).

Define (for this paragraph only)

$$\psi(x) = \Pr\{\exists t \geqq 0 \ni Z_t^x \notin O^R\}.$$

The complement of O^R has a non-empty interior in E^m. Using standard probabilistic arguments it can be shown from this and the definition of $\{Z_t\}$ that for all x, $\psi(x) > 0$. ψ is a continuous function. For $x \in K^m$ with $|x| \geqq R$, $\psi(x) = 1$. It follows that

$$(4.2.6) \qquad a = \inf_{\{x\,:\,x\in K^m,\ |x|<R\}} \psi(x) > 0.$$

(4.2.5) and the Markov property of Z_t^x imply that $\psi(x) \geqq \psi(x)+(1-a)a$ for all $x \in K^m$, $|x| < R$. Then for all $x \in E^m$ (4.2.6) yields

$$\Pr\{\exists t < \infty : Z_t^x \notin O^R\} = 1.$$

That is to say, (4.2.1) is satisfied, which was to be proved.

LEMMA 4.2.2. *Let k_{ij} be piecewise differentiable functions on E^m satisfying*

$$0 \leqq k_{ij} \leqq 1$$

$$k_{ij}(x) = 1 \qquad |x| \leqq 1$$

for all $i = 2, 3, \cdots, j = 0, 1, \cdots$. *Let* ε_i, $i = 1, 2, \cdots$, *satisfy* $\lim_{i\to\infty} \varepsilon_i = 0$. *Suppose* k_{ij} *satisfies the Euler equations*—(4.1.8)—*on*

$$S_{ij} = \{x: 1 < |x| < i, \quad \text{or} \quad x \notin K^m \quad \text{and} \quad i \leqq |x| < i+j\}$$

and there is a sequence l_i *such that*

$$k_{ij}(x) = l_i(x) \leqq \varepsilon_i \qquad \text{for } x \in K^m, |x| \geqq i.$$

Then

$$\lim_{i\to\infty} \lim_{j\to\infty} k_{ij}(x) = \mathscr{K}(x) \tag{4.2.7}$$

where the above limit exists and for each i $\lim_{j\to\infty} k_{ij}(x)$ *is uniform on all compact sets* C *such that* $C \subset O^i$, *and* $\lim_{j\to\infty} k_{ij}(x) \to \mathscr{K}(x)$ *uniformly on all compact sets* $C \subset \{x: |x| > 1\}$. ($\mathscr{K}$ *is defined by* (4.1.4).) [*Note*: $\lim_{j\to\infty} S_{ij} = O^i$.]

PROOF. For this proof only, define $T_{ij}^x = \inf\{t: Z_t^x \notin S_{ij}\}$ and $\tau_{ij}^x(t) = \inf(t, T_{ij}^x)$. Since S_{ij} is bounded, $T_{ij}^x < \infty$ with probability one. Since k_{ij} satisfies the Euler equations on S_{ij}, $k_{ij}(Z^x_{\tau_{ij}^x(t)})$ is a Martingale. This is immediately seen from the fact that the operator on the left of (4.1.8) is the generator of the diffusion for $x \in S_{ij}$ (see e.g. Itô and McKean ((1965) page 304) or Dynkin ((1965) page 159). Thus for $x \in S_{ij}$

$$k_{ij}(x) = E(k_{ij}(Z^x_{T_{ij}^x})). \tag{4.2.8}$$

For this proof only, define the random variables

$$\sigma_{ij}^x = \sup_{0 \leqq t \leqq T_{ij}^x} |Z_t^x|.$$

Lemma 4.2.1 and the continuity of the diffusion paths guarantee that $\sigma_i^x < \infty$ with probability 1. It follows that for each fixed $x \in S_{ij}$

$$\Pr\{Z^x_{T_{ij}^x} \in K^m\} + \Pr\{|Z^x_{T_{ij}^x}| = 1\} \to 1$$

as $j \to \infty$ with each term on the left having a limit as $j \to \infty$; and, further, that $Z^x_{T_{ij}^x}$ has a limiting distribution as $j \to \infty$. Thus, for all $x \in S_{ij}$ $\lim_{j\to\infty} k_{ij}(x)$ exists and satisfies

$$\lim_{j\to\infty} \Pr\{|Z^x_{T_{ij}^x}| = 1\} \leqq \lim_{j\to\infty} k_{ij}(x) \leqq \varepsilon_i + (1-\varepsilon_i) \lim_{j\to\infty} \Pr\{|Z^x_{T_{ij}^x}| = 1\} \tag{4.2.9}$$

Let C be a compact set such that $C \subset O^i$. For all j sufficiently large, say $j > J_i$, $\{x: x = y+z, y \in C, |z| < J^{-1}\} \subset S_{ij} \cup \{x: |x| \leqq 1\}$. Using Harnack's inequality it follows that there is a bound b, say (depending on C), such that $|\nabla k_{ij}(x)| < b$ for all $x \in C$ and $j > J_i$. Since each k_{ij} is a continuous function it follows that the limit $\lim_{j\to\infty} k_{ij}(x)$ is uniform on C and that the functions $\lim_{j\to\infty} k_{ij}(x)$ are uniformly continuous on C.

For this paragraph define $T^x = \inf\{t: |Z_t^x| \leqq 1\}$. (Note that $\Pr\{T^x < \infty\} = \mathscr{K}(x)$.), and let $\sigma^x = \sup_{0 \leqq t \leqq T^x} |Z_t^x|$. Again note that $\Pr\{\sigma^x < \infty\} \doteq \mathscr{K}(x)$, so that $\lim_{r\to\infty} \Pr\{\sigma^x < r\} = \mathscr{K}(x)$. It follows from this that

$$\lim_{i\to\infty} \lim_{j\to\infty} \Pr\{|Z^x_{T_{ij}^x}| = 1\} = \mathscr{K}(x).$$

Hence from (4.2.9)

$$\lim_{i\to\infty}\lim_{j\to\infty} k_{ij}(x) = \mathscr{K}(x). \tag{4.2.10}$$

If $C \subset \{x: |x| > 1\}$ is a compact set then for all $i > I$, say, $C \subset O^i$. From the uniform continuity on C of $\lim_{j\to\infty} k_{ij}(x)$ which was established above it follows that $\lim_{j\to\infty} k_{ij}(x) \to \mathscr{K}(x)$ uniformly on C. This completes the proof of the lemma.

LEMMA 4.2.3. *Suppose $\{Z_t\}$ is recurrent. Let $k_i \in J$, $i = 2, 3, \cdots$, be a sequence such that $k_i(x) = 1$ for $|x| \leqq 1$, $k_i(x) = 0$ for $|x| \geqq i$, and k_i satisfies* (4.1.8) *on $\{x: 1 < |x| < i\}$. Then if $C \subset E^m$ is a compact set $k_i(x) \to 1$ uniformly on C.*

PROOF. The proof is similar to that of Lemma 4.2.2, but somewhat simpler. For this proof, define

$$U_i^x = \inf\{t: |Z_t^x| \leqq 1 \quad \text{or} \quad |Z_t^x| \geqq i\}.$$

$\{k_i(Z_t^x)\}$ is a martingale for $0 \leqq t \leqq U_i^x$. Hence for $1 < |x| < i$

$$k_i(x) = \Pr\{|Z^x_{U_i^x}| = 1\}.$$

Since $\{Z_t\}$ is recurrent $\Pr\{|Z^x_{U_i^x}| = 1\} \to 1$ as $i \to \infty$. Hence for each fixed x $k_i(x) \to 1$ as $i \to \infty$. By Harnack's inequality, for all i sufficiently large $\sup_{x\in C} |\nabla k_i(x)| < \infty$. Hence $k_i(x) \to 1$ uniformly on C which was to be proved.

4.3. *The characterization.* We are now in a position to prove the main theorem of this chapter. As noted, this theorem gives an analytic criterion, in terms of f^*, for deciding whether $\{Z_t\}$ is transient or recurrent. As indicated in Sub-section 1.3 this analytic criterion plays a key role in the proof in Section 5. Corollaries 4.3.2–4.3.4 and the results in Section 6 give applications of Theorem 4.3.1.

Throughout this section we let $k_0 = \int |\nabla\mathscr{K}(x)|^2 f^*(x)\,dx$ where $\mathscr{K}$ is defined by (4.1.4). The set J, used below, is defined following (4.1.2).

THEOREM 4.3.1. *$\{Z_t\}$ is recurrent if and only if*

$$\inf_{j\in J} \int |\nabla j(x)|^2 f^*(x)\,dx = 0. \tag{4.3.1}$$

If $\{Z_t\}$ is transient then

$$\inf_{j\in J} \int |\nabla j(x)|^2 f^*(x)\,dx \geqq k_0 > 0. \tag{4.3.2}$$

$\{Z_t\}$ is recurrent if and only if there is a sequence $k_i \in J$ such that $k_i(x) = 0$, $|x| \geqq i$ and

$$\lim_{i\to\infty} \int |\nabla k_i(x)|^2 f^*(x)\,dx = 0. \tag{4.3.3}$$

[*Note*: If $\{Z_t\}$ is transient and $\mathscr{K} \in J$ then we clearly have equality in (4.3.2). However, if $\mathscr{K} \notin J$, which is possible, then it seems likely that there may be strict inequality in (4.3.2).]

PROOF. We begin with the transient case. Suppose there is a piecewise differentiable function $k \in J$ such that

$$\int |\nabla k(x)|^2 f^*(x)\,dx = k_0 - \varepsilon' < k_0. \tag{4.3.4}$$

Since $k'(x) = \sup(k(x), 1) \in J$ and satisfies $\int |\nabla k'(x)|^2 f^*(x)\,dx \leqq \int |\nabla k(x)|^2 f^*(x)\,dx$ we may assume without loss of generality that $k(x) = 1$, $|x| \leqq 1$. We will eventually deduce a contradiction and thus show that no such function k can exist.

Given k as above define $k_{ij}(x)$, $i, j = 1, 2, \cdots$, as follows: As in Lemma 4.2.2 let

$$S_{ij} = \{x : 1 < |x| < i, \quad \text{or} \quad x \notin K^m \quad \text{and} \quad i \leqq |x| < i+j\}.$$

Let $k_{ij}(x)$ be the unique continuous function satisfying the Euler equations—(4.1.8)—on S_{ij} and

$$\begin{aligned} k_{ij}(x) &= 1 & |x| &\leqq 1 \\ k_{ij}(x) &= k(x) & |x| &> 1, x \notin S_{ij}. \end{aligned}$$

From the minimization properties of functions satisfying the Euler equations it is evident that

$$\int |\nabla k_{ij}(x)|^2 f^*(x)\,dx \leqq \int |\nabla k(x)|^2 f^*(x)\,dx \leqq k_0 - \varepsilon'. \tag{4.3.5}$$

Choose R so that

$$\int_{|x|<R} |\nabla \mathscr{K}(x)|^2 f^*(x)\,dx > k_0 - \varepsilon'/2. \tag{4.3.6}$$

It follows from Lemma (4.2.2) there is a sequence $j(i)$ such that $k_{ij(i)}(x) \to \mathscr{K}(x)$ uniformly for $|x| = R$.

To save writing subscripts define $k_i(x) = k_{ij(i)}(x)$.

It follows from Harnack's inequality that there is a bound b', say, such that for $i \geqq R+1$, $|x| = R$,

$$|\nabla k_i(x)| < b' < \infty \quad \text{and} \quad |\nabla \mathscr{K}(x)| < b'. \tag{4.3.7}$$

For any two twice continuously differentiable functions l, m, say, on $1 < |x| < R$, each continuous at $|x| = 1$ and $|x| = R$ and each satisfying the Euler equation—(4.1.8)—on $\{x : 1 < |x| < R\}$ the appropriate version of the general form of Gauss' divergence theorem is

$$\begin{aligned} &\int_{1<|x|<R} \nabla l(x) \cdot \nabla m(x) f^*(x)\,dx \\ &\qquad = \int_{|x|=1,\,|x|=R} l(x)(\nabla m(x) \cdot n(x)) f^*(x)\,ds \end{aligned} \tag{4.3.8}$$

where $n(x)$ is the unit normal at x to the surface of $\{x : 1 < |x| < R\}$ in the outward direction and ds represents the usual differential appropriate for the indicated surface integral.

If in (4.3.8) we let $l(x) = \mathscr{K}(x) - k_i(x)$ and $m(x) = \mathscr{K}(x) + k_i(x)$ we have

$$\begin{aligned} &\int_{1<|x|<R} (|\nabla \mathscr{K}(x)|^2 - |\nabla k_i(x)|^2) f^*(x)\,dx \\ &\quad = \int_{1<|x|<R} \nabla(\mathscr{K}(x) - k_i(x)) \cdot \nabla(\mathscr{K}(x) + k_i(x)) f^*(x)\,dx \\ &\quad = \int_{|x|=1,\,|x|=R} (\mathscr{K}(x) - k_i(x))(\nabla(\mathscr{K}(x) + k_i(x)) \cdot n(x)) f^*(x)\,ds. \end{aligned} \tag{4.3.9}$$

For $|x| = 1$, $k_i(x) = \mathscr{K}(x)$, hence the only contribution on the right of (4.3.9) is from the integral over the surface $|x| = R$. From (4.3.7) it follows that

$$\nabla(\mathscr{K}(x)+k_i(x))\cdot n(x) \leqq 2b' < \infty.$$

Hence using (4.3.7) in the remaining part of the integral on the right of (4.3.9) yields

$$\int_{1<|x|<R}(|\nabla\mathscr{K}(x)|^2-|\nabla k_i(x)|^2)f^*(x)\,dx \to 0 \qquad \text{as} \quad i\to\infty. \tag{4.3.10}$$

(4.3.10) contradicts (4.3.5)–(4.3.6) (since $k_i(x) = k_{ij(i)}(x)$). Hence (4.3.4) must be false. We have thus established that

$$\inf_{j\in J}\int |\nabla j(x)|^2 f^*(x)\,dx \geqq k_0$$

which is (4.3.2).

It remains only to consider the recurrent case. If a sequence $\{k_i\} \subset J$ exists satisfying (4.3.3), etc. then by the above $\{Z_t\}$ is recurrent. For the converse, assume $\{Z_t\}$ is recurrent. For the remainder of the proof let k_i', $i = 1, 2, 3, \cdots$, be the continuous functions satisfying (4.1.8) on $\{x: 1 < |x| < i\}$ and

$$k_i'(x) = 1 \qquad |x| \leqq 1$$

$$k_i'(x) = 0 \qquad |x| \geqq i.$$

(The k_i' are uniquely defined.) Let $\mathbf{1}(x)$ be the function which is identically 1 on E^m. Fix $r < \infty$. By Lemma 4.2.3 $k_i(x) \to 1$ uniformly for $|x| = r$. Hence as in (4.3.8)–(4.3.10) for any $r < \infty$

$$\begin{aligned}\int_{1<|x|<r}&|\nabla k_i(x)|^2 f^*(x)\,dx\\ &= \int_{1<|x|<r}(|\nabla k_i(x)|^2-|\nabla \mathbf{1}(x)|^2)f^*(x)\,dx\\ &= \int_{|x|=r}(k_i(x)-1)(\nabla k_i(x)\cdot n(x))f^*(x)\,ds \to 0 \qquad \text{as} \quad i\to\infty.\end{aligned} \tag{4.3.11}$$

By the minimizing property of solutions of (4.1.8) it follows that $\int |\nabla k_i(x)|^2 f^*(x)\,dx$ is decreasing as $i \to \infty$. Suppose

$$\lim_{i\to\infty}\int |\nabla k_i(x)|^2 f^*(x)\,dx = \varepsilon > 0. \tag{4.3.12}$$

For this proof let α be an integer sufficiently large so that

$$\int |\nabla k_\alpha(x)|^2 f^*(x)\,dx < 3\varepsilon/2$$

and let $\beta > \alpha$ be sufficiently large so that

$$\int_{1<|x|<\alpha} |\nabla k_\beta(x)|^2 f^*(x) < \varepsilon/8.$$

(The existence of β is guaranteed by (4.3.11).) Let $k'(x) = (k_\alpha(x)+k_\beta(x))/2$. Since $k'(x) = 1$, $|x| \leqq 1$ and $= 0$ $|x| \geqq \beta$ the minimal property of k_β implies

$$\int |\nabla k_\beta(x)|^2 f^*(x)\,dx \leqq \int |\nabla k'(x)|^2 f^*(x)\,dx. \tag{4.3.13}$$

Using Cauchy–Schwarz and the fact that $k_\alpha(x) = 0$ for $|x| > \alpha$

$$\begin{aligned}
&\int |\nabla k'(x)|^2 f^*(x)\,dx \\
&\quad = (\tfrac{1}{4}) \int |\nabla k_\alpha(x) + \nabla k_\beta(x)|^2 f^*(x)\,dx \\
&\quad \leqq (\tfrac{1}{4}) \int |\nabla k_\alpha(x)|^2 f^*(x)\,dx \\
&\qquad + (\tfrac{1}{2}) \int_{|x|<\alpha} |\nabla k_\alpha(x)\cdot \nabla k_\beta(x)|\, f^*(x)\,dx \\
&\qquad + (\tfrac{1}{4}) \int |\nabla k_\beta(x)|^2 f^*(x)\,dx \\
&\quad \leqq (\tfrac{2}{4})\cdot(3\varepsilon/2) + (\tfrac{1}{2})(\textstyle\int_{|x|<\alpha} |\nabla k_\alpha(x)|^2 f^*(x)\,dx)^{\frac{1}{2}} \\
&\qquad \cdot (\textstyle\int_{|x|<\alpha} |\nabla k_\beta(x)|^2 f^*(x)\,dx)^{\frac{1}{2}} \\
&\quad < \varepsilon.
\end{aligned} \tag{4.3.14}$$

(4.3.14) contradicts (4.3.12). Thus $\lim_{i\to\infty} \int |\nabla k_i(x)|^2 f^*(x)\,dx = 0$ which proves the assertion at (4.3.3). (4.3.1) is an immediate consequence of (4.3.3) and (4.3.2). This completes the proof of the theorem.

The result below could also have been deduced directly from Lemma 4.2.1, but we prefer the following proof since it gives a simple application of Theorem 4.3.1.

COROLLARY 4.3.2. *If K is compact then $\{Z_t\}$ is recurrent.*

PROOF. For this proof let $\rho = \sup_K |x| < \infty$. Also, $F(K) < \infty$. For $|x| > \rho$

$$f^*(x) \leqq F(K) \exp(-(|x|-\rho)^2/2)/(2\pi)^{m/2}. \tag{4.3.15}$$

Let $j_i(x) = 1$, $|x| \leqq i$, $\exp(-(|x|-i))$ for $|x| > i$. Then $j_i(x) \in J$, and for $i \geqq \rho$ we have

$$\begin{aligned}
&\int |\nabla j_i(x)|^2 f^*(x)\,dx \\
&\quad \leqq (F(K)/(2\pi)^{m/2}) \int_{|x|>i} \exp(-2(|x|-i)) \exp(-(|x|-\rho)^2/2)\,dx \to 0 \\
&\qquad \text{as } i \to \infty.
\end{aligned} \tag{4.3.16}$$

(4.3.16) and (4.3.1) establish that $\{Z_t\}$ is recurrent.

We can use one consequence of Theorem 4.3.1 in the proof of Lemma 5.4.1. Applications of the following corollary of statistical interest are presented in Section 6.

COROLLARY 4.3.3. *Suppose there are positive constants b_0, b_1, b_2 such that for $|x| > b_0$*

$$f^*(x) \geqq b_1 |x|^{2-m+b_2}. \tag{4.3.17}$$

Then $\{Z_t\}$ is transient.

PROOF. Since, from Lemma 3.1.4, $f^*(x) \leqq \exp(-d^2(x)/2) f^*(\pi(x))$ it can be deduced from (4.3.17) that $K = E^m$.

Let $b_3 = \inf_{|x|<b_0} f^*(x) > 0$ and let $b_4 = \min(b_1, b_3/b_0{}^{2-m+b_2})$. Then $f^*(x) \geqq b_4|x|^{2-m+b_2}$ for $|x| > 1$. For any $j \in J$

$$(4.3.18) \qquad \int |\nabla j(x)|^2 f^*(x)\, dx \geqq b_1 \int_{|x|>1} \left(\frac{\partial}{\partial r} j(x)\right)^2 |x|^{2-m+b_2}\, dx$$

where $r = |x|$. Let (r, φ) denote spherical co-ordinates on E^m with $r = |x|$ and $dx = \omega_m(\varphi) r^{m-1}\, dr\, d\varphi$. For simplicity we let $k(r, \varphi) = j(x(r, \varphi))$. We rewrite (4.3.18) as

$$(4.3.19) \qquad \int |\nabla j(x)|^2 f^*(x)\, dx \geqq b_4 \int \omega_m(\varphi)\, d\varphi \int_{r>1} \left|\frac{\partial}{\partial r} k(r, \varphi)\right|^2 r^{1+b_2}\, dr.$$

The infimum of the right-hand side of (4.3.19) over all k such that $k(1, \varphi) = 1$ and $\lim_{r\to\infty} k(r, \varphi) = 0$ may be explicitly computed from the appropriate Euler equation (which here is an ordinary differential equation). The minimizing choice for k is $k(r, \varphi) = 1$ $|r| \leqq 1$ and $k(r, \varphi) = b_5 \int_r^\infty s^{-(1+b_2)}\, ds$ for $|r| > 1$ where $b_5 = (\int_1^\infty s^{-(1+b_2)}\, ds)^{-1} > 0$. We have

$$(4.3.20) \qquad \inf_{j\in J} \int |\nabla j(x)|^2 f^*(x)\, dx \geqq b_4 b_5{}^2 \int_1^\infty s^{-(1+b_2)}\, ds = b_4 b_5 > 0.$$

The result of Theorem 4.3.1 immediately completes the proof of the corollary.

In a similar manner we can prove the following, some applications of which are also given in Chapter 6.

COROLLARY 4.3.4. *Suppose there are positive constants b_1 and b_2 such that $f^*(x) \leqq b_1|x|^{2-m-b_2}$. Then $\{Z_t\}$ is recurrent.*

PROOF. Let

$$\begin{aligned} j_i(x) &= 1 && |x| \leqq 1 \\ &= 0 && |x| \geqq i \\ &= (r^{b_2}-1)/(i^{b_2}-1) && 1 < |x| < i. \end{aligned}$$

It can then be computed that

$$\int |\nabla j_i(x)|^2 f^*(x)\, dx \to 0 \qquad \text{as } i \to \infty$$

which implies that $\{Z_t\}$ is recurrent.

4.4. *A mean value representation.* In equations (5.7.9)–(5.7.11) we will need a representation of the form $j(x) = \int j(y) q_x(y)\, dy$ for certain solutions of the differential equation (4.1.8) in the case where $|\gamma_F(x)|$ is bounded and $\{Z_t\}$ is recurrent. Here, q_x is a probability density on the ball of radius one with center x. We will also need certain additional facts about the functions q_x, as described in Lemma 4.4.2 below. We remind the reader that (4.1.8) may be written $\nabla\cdot(f^* \times \nabla j) = 0$.

One minor complication in the following is that we need the above representation for all x whereas we will eventually only be able to assume j satisfies (4.1.8) for $|x| > 1$ and for $|x| < 1$. For this reason we first prove a result of the desired type for the general case where (4.1.8) is satisfied everywhere in the ball of radius $\frac{3}{4}$ about x. This is Lemma 4.4.1. Then we prove the lemma we need for our special case.

The following Lemma is a special case of a result due to J. Bramble ((1963), and personal communication).

LEMMA 4.4.1. *Suppose* $|\nabla f^*/f^*| \leqq B < \infty$. *Then there is a* $C < \infty$, *depending only on* B *and* m *(not on* f^**) and a family of probability densities* $\{r_x\}_{x \in E^m}$ *on the ball of radius* $\frac{1}{2}$ *with center* x *such that* $r_x < C$ *and*

$$j(x) = \int j(y) r_x(y)\, dy \tag{4.4.1}$$

for all j *satisfying the equation* $\nabla \cdot (f^* \times \nabla j) = 0$ *on* $\{y : |y-x| < \frac{3}{4}\}$.

PROOF. Let $\rho = \min(1/4m, 1/8B)$. For $|x-y| < \rho$ let

$$\begin{aligned} s(x,y) = \omega(x)(x-y)\cdot \nabla f^*(y)/|x-y|^m + (y-x)\cdot \nabla f^*(y)/|x-y|^{m-\frac{1}{2}}\rho^{\frac{1}{2}} \\ + f^*(y)/2|x-y|^{m-\frac{1}{2}}\rho^{\frac{1}{2}})/f^*(x). \end{aligned} \tag{4.4.2}$$

For $|x-y| \geqq \rho$ define $s(x,y) = 0$. In (4.4.2) $\omega(x)$ is chosen so that

$$\int s(x,y)\, dy = 1. \tag{4.4.3}$$

Note that since $|(y-x)\cdot \nabla f^*(y)| \leqq |y-x|\, |\nabla f^*(y)|$ we have

$$\begin{aligned} s(x,y) \geqq \omega(x)(\tfrac{1}{2}|x-y|^{m-\frac{1}{2}}\rho^{\frac{1}{2}} - B/|x-y|^{m-1} \\ - B/|x-y|^{m-\frac{3}{2}}\rho^{\frac{1}{2}}) f^*(y)/f^*(x). \end{aligned} \tag{4.4.4}$$

Observing that $|x-y| < \rho \leqq \frac{1}{8}B$ we immediately see that $s \geqq 0$. Noting further that $f^*(y)/f^*(x) \geqq \exp(-B|y-x|)$ we also see that there is a bound ω', say, depending only on B and m such that $\omega(x) < \omega'$.

It is shown in Bramble (1963)[2] that for $m \geqq 2$ for any j satisfying (4.1.8) on $\{y : |x-y| < \rho\}$

$$j(x) = \int j(y) s(x,y)\, dy. \tag{4.4.5}$$

This result is also valid for $m = 1$. Since s is not bounded it is not sufficient for our purposes to define $r_x(y) = s(x,y)$. However, we may let

$$r_x(y) = \int \cdots \int s(x,t_1)s(t_1,t_2) \cdots s(t_{2m},y) \textstyle\prod_{i=1}^{2m} dt_i. \tag{4.4.6}$$

---

[2] In brief, Bramble's argument for the case $m \geqq 3$ is as follows. Let

$$\Gamma(x,y) = (\omega(x)/(m-2))(|x-y|^{2-m} + \rho^{2-}/2^{\alpha}\alpha - (m-2)/\alpha|x-y| \text{-} \rho^{\frac{1}{2}})$$

where $\alpha = (2m-5)/2$. With $\omega(\cdot)$ as above, Γ is a "parametrix" (see, e.g. Miranda (1955)) of the operator $\nabla \cdot (f\mathrm{P}^*\nabla)$ (operating with respect to y) and satisfies $\Gamma > 0$ for $0 < |y-x| < \rho$ and $\nabla\Gamma \cdot (y-x) = 0$ on the circle $|y-x| = \rho$. The operator isself adjoint, and $s(x,y) = \nabla \cdot (f^*\nabla\Gamma)$. It follows by standard techniques using the above properties of Γ that (4.4.5) is satisfied.

Since $s(t, \cdot)$ is a probability density, so is $r_x(\cdot)$. Since $\rho \leqq 1/4m$, $r_x(y) = 0$ if $|y-x| > \frac{1}{2}$. We note that

(4.4.7) $$s(x, y) \leqq \omega(x)(2B+1)\, e^{B\rho}/|x-y|^{m-\frac{1}{2}}\rho^{\frac{1}{2}}.$$

The derivation of this inequality is similar to that of (4.4.4). Note that $\omega(x) < \omega'$. Hence

(4.4.8) $$r_x(y) \leqq (\omega'(2B+1)\, e^{\frac{1}{8}}/\rho^{\frac{1}{2}})^{2m+1}$$

$$\int \cdots \int_{|x-t_1| \leqq \rho,\ |t_{i+1}-t_i| \leqq \rho,\ |y-t_{2m}| \leqq \rho} |x-t_1|^{\frac{1}{2}-m}|t_1-t_2|^{\frac{1}{2}-m} \cdots |t_{2m}-y|^{\frac{1}{2}-m} \prod_{i=1}^{2m} dt_i.$$

The right side of (4.4.8) is bounded by a constant independent of x, y. Hence there is a $C < \infty$ such that $r_x(y) < C$ for all x, y. Finally it is possible to compute directly from (4.4.5) and (4.4.6) that

$$j(x) = \int j(y) r_x(y)\, dy.$$

This is the desired result, and the proof is complete.

We now derive the exact representation we need.

LEMMA 4.4.2. *Suppose* j, $j \leqq 1$, *is a continuous function, twice continuously differentiable on* $\{x: |x| > 1\}$ *and satisfying* $j(x) \equiv 1$ *for* $|x| \leqq 1$, *and* $\nabla \cdot (f^* \nabla j) = 0$ *on* $\{x: |x| > 1\}$. *Suppose* $|\nabla f^*/f^*| < B'$. *Then there is a* $C' < \infty$ *depending only on* B' *and* m (*but not on* j *or* f^*) *and a family of probability densities* $\{q_x\}$ *on the ball of radius one with center* x *such that* $q_x < C'$ *and*

(4.4.9) $$j(x) = \int j(y) q_x(y)\, dy.$$

[Note: For $|x| < \frac{3}{2}$ the choice of q_x may depend on j.]

PROOF. Let $\{r_x\}$ be as in Lemma 4.4.1. From the properties of j described above it follows immediately that j is a superharmonic function. Hence it is reasonable that

(4.4.10) $$j(x) \geqq \int j(y) r_x(y)\, dy$$

for all $x \in E^m$. Rather than follow this line of reasoning further, we use below a more explicit method to establish (4.4.10).

For this paragraph fix x, $1 < |x| < \frac{3}{2}$. For $|z| = 1$ let $v_j(z)$ denote the value of the normal derivative of j to the surface of the sphere $\{z: |z| \leqq 1\}$ from the *outside* of the sphere. Since $j \leqq 1$ everywhere and $j \equiv 1$ on the sphere $\{z: |z| \leqq 1\}$ we see that $v_j(z) \geqq 0$ for all $z: |z| = 1$ (that is, $\nabla^+ j(z) \cdot z \leqq 0$ where

$$\nabla^+ j(z) = \lim_{x \to z, |x| > |z|} \nabla j(x)$$

when these concepts are well defined). Possibly $v_j(z) = \infty$. In Bramble ((1963) (2.7)) it is shown that since j satisfies $\nabla \cdot (f^* \times \nabla j) = 0$ on $\{x: |x| > 1\}$ and $\{x: |x| < 1\}$

$$j(x) = \int j(y) r_x(y)\, dy + \int_{|y|=1} v_j(y) \Gamma_x(y)\, dy$$

where $\Gamma_x(y) \geqq 0$. (In Bramble (1963) Γ is constructed so that $f^*\nabla^2\Gamma + \nabla f^* \cdot \nabla\Gamma = r_x$.) Hence, (4.4.10), $j(x) \geqq \int j(y) r_x(y)\, dy$, for $1 < |x| < \frac{3}{2}$. (4.4.10) is obviously valid for $|x| \geqq \frac{3}{2}$ (with equality), and it is also valid for $|x| \leqq 1$ since $j \leqq 1$.

For this proof only let $L_x = \{y : |y| \leqq 1, |x-y| \leqq 1\}$ and for $|x| < \frac{3}{2}$ let

$$\begin{aligned} \rho_x(y) &= 1/\textstyle\int_{L_x} dy \qquad & y \in L_x \\ &= 0 & \text{otherwise.} \end{aligned}$$

For $|x| < \frac{3}{2}$, $\rho_x(y) \leqq 1/\int_{L_z} dy < \infty$ where $|z| = \frac{3}{2}$. For $|x| < \frac{3}{2}$

$$1 = \int j(y) \rho_x(y)\, dy \geqq j(x). \tag{4.4.11}$$

(4.4.10) and (4.4.11) together imply the existence of an α_x $0 \leqq \alpha_x \leqq 1$ such that, for $|x| < \frac{3}{2}$

$$j(x) = \int j(y)(\alpha_x r_x(y) + (1-\alpha_x)\rho_x(y))\, dy. \tag{4.4.12}$$

Define

$$\begin{aligned} q_x(y) &= r_x(y) & |x| \geqq \tfrac{3}{2} \\ &= \alpha_x r_x(y) + (1-\alpha_x)\rho_x(y) & |x| < \tfrac{3}{2}. \end{aligned}$$

q_x satisfies (4.4.9). Since $q_x(y) \leqq \sup(r_x(y), \rho_x(y))$ for $|x| < \frac{3}{2}$ it is possible to find a C' such that $q_x(y) < C'$ for all x, y. This completes the proof.

5. Statement and proof of the main theorem.

5.1. *Statement of the theorem.* To facilitate the division of the proof we state the theorem in two parts. The first is our necessary condition for admissibility, the second is our sufficient condition.

Before the statement, we remind the reader that $\{Z_t\}$ is the diffusion in E^m with local mean $\nabla f^*/f^* = \nabla(\log f^*)$ and local variance $2I$. Section 4 presents several relevant properties of this diffusion, including a criteria for deciding whether $\{Z_t\}$ is transient or recurrent (see Sub-section 4.3). Examples concerning special cases and other related results using Theorem 5.1.1 can be found in Section 6. We also remind the reader that the set K is the closed convex hull of the support of F, which is also given by $\{y + \nabla f^*(y)/f^*(y) : y \in E^m\}$ (see Lemma 3.2.1 and the remark following it).

THEOREM 5.1.1. *A necessary condition for δ to be admissible is that there exist a nonnegative measure F such that $f^* < \infty$ and $\delta(x) = \delta_F(x)$ for almost all x in E^m (with respect to Lebesgue measure). Furthermore*

(A) *If $\{Z_t\}$ is transient then δ is inadmissible*

(B) *If $\{Z_t\}$ is recurrent and*

$$\sup_{x \in K} |\gamma_F(x)| = B < \infty \tag{5.1.1}$$

then δ is admissible.

Note that Theorem 3.3.1 establishes that (5.1.1) is equivalent to

$$\sup_{\theta \in K} R(\theta, \delta) < \infty.$$

Before proceeding further we remark that the first sentence of Theorem 5.1.1 is a restatement of Theorem 3.1.1 and hence need not be reproved here.

5.2. *Proof of Part A.* The proof of Part A of Theorem 5.1.1 is essentially contained in Sections 3 and 4 as outlined in the introductory Sub-section 1.3. (For simplicity Sub-section 1.3 only considered the case where $K = E^m$.) We review the main steps below. We begin by assuming that $\{Z_t\}$ is transient.

Stein's necessary condition for admissibility states that $\delta = \delta_F$ (a.e.) is admissible only if there is a sequence of nonnegative finite measures G_i satisfying the conditions $\mathscr{A}_1': G_i(\{0\}) = 1$, and $\mathscr{B}_1: G_i$ has compact support, and such that

$$\int \|\nabla \hat{j}_i(x)\|^2 f^*(x)\, dx \to 0 \tag{5.2.1}$$

where $\hat{j}_i(x) = (g_i^*(x)/f^*(x))^{\frac{1}{2}}$. This result is contained in Sub-section 1.3 through (1.3.4). It is also remarked in Sub-section 1.3 that without loss of generality we may assume $\mathscr{A}_2: \hat{j}_i(x) \geqq 1$ for $|x| = 1$ in place of $\mathscr{A}_1'$ and we will do so here. $\hat{j}$ is clearly a continuously differentiable function.

Lemma 3.5.1. states that condition $\mathscr{B}_1$ implies the condition

$$\mathscr{B}_2': \lim_{r\to\infty} \sup_{\{x\,:\, x\in K^m,\ |x|=r\}} \hat{j}_i(x) = 0.$$

We remark that since $\{Z_t\}$ is transient K is not compact (Corollary 4.3.2.). Hence for all sufficiently large r, $K^m \cap \{x: |x| = r\}$ is not empty, and the condition $\mathscr{B}_2'$ is not an empty one.

We established in Theorem 4.3.1 that since $\{Z_t\}$ is transient there is a $k_\circ > 0$ such that for all continuously differentiable j the conditions $\mathscr{A}_3: j(x) \geqq 1$ $|x| = 1$; and $\mathscr{B}_3: \lim_{r\to\infty} \sup_{K^m \cap \{|x|=r\}} j(x) = 0$ imply

$$\int \|\nabla j(x)\|^2 f^*(x)\, dx \geqq k_\circ > 0.$$

Hence it is impossible for there to exist a sequence G_i satisfying $\mathscr{A}_1'$, $\mathscr{B}_1$, and (5.2.1). This establishes that δ_F cannot be admissible. The proof of Part A of the theorem is complete.

5.3. *Construction of a "smooth" recurrent diffusion.* We now begin the proof of Part B of the theorem. This section contains a lemma of a rather technical nature which is needed only in the case where $K \neq E^m$. The proof in Sub-section 5.7 is valid and may be understood in the case $K = E^m$ without the result of this section.

We need the result of this section for the following technical reason: If $K \neq E^m$ then $\nabla f^*(x)/f^*(x)$ is not bounded. Hence given $k < \infty$ there is no bound k_1, say, such that $|y-x| < k$ implies $f^*(y)/f^*(x) < k_1$ *uniformly* in x, y. However, as we have constructed our proof we need a bound of this type at the step (5.7.12). Also, as we have constructed our proof we need a similar smoothness result for the functions j_i which appear in (1.3.6). The result of this section is a useful tool for proving the existence of such smooth j_i when $K \neq E^m$, which will be done in the next section. Part B of the theorem can probably be proved without either of these technical results, but we have not found a shorter or more direct alternative proof.

Given F, satisfying (5.1.1) we will construct another generalized prior satisfying the conditions described in the lemma below. We call this prior F_1, and we denote the diffusion corresponding to F_1 by $\{Z_t^1\}$ and the estimator by $x+\gamma_1(x) = x+\nabla f_1^*(x)/f_1^*(x)\cdot F_1 \geqq F$ means that for all measurable $A \subset E^m$, $F_1(A) \geqq F(A)$.

LEMMA 5.3.1. *Suppose F is such that $\{Z_t\}$ is recurrent and* (5.1.1) *is satisfied. Then there exists a measure F_1 such that $F_1 \geqq F$, $\sup_{x\in E^m} |\gamma_1(x)| \leqq B_1 < \infty$, (hence $K_{F_1} = E^m$), and $\{Z_t^1\}$ is recurrent.*

PROOF. Define

$$F_1(d\theta) = F(d\theta)+f^*(\pi(\theta))\, e^{-d(\theta)}\, d\theta. \tag{5.3.1}$$

Clearly $F_1 > F$ and $K_{F_1} = E^m$.

A general geometric fact is that for $\theta_1, \theta_2 \in E^m$

$$|d(\theta_1)-d(\theta_2)| \leqq |\theta_1-\theta_2|$$

and

$$|\pi(\theta_1)-\pi(\theta_2)| \leqq |\theta_1-\theta_2|.$$

Hence as $|\theta_2-\theta_1| \to 0$

$$\begin{aligned} &f^*(\pi(\theta_1)e^{-d(\theta_1)}-f^*(\pi(\theta_2))e^{-d(\theta_2)} \\ &\quad = f^*(\pi(\theta_1)\,(e^{-(d\theta_1)}-e^{-d(\theta_2)})+e^{-d(\theta_2)}(f^*(\pi(\theta_1))-f^*(\pi(\theta_2))) \\ &\quad \leqq f^*(\pi(\theta_1))e^{-d(\theta_1)}|\theta_2-\theta_1|+e^{-d(\theta_2)}f^*(\pi(\theta_1))(e^{+B|\theta_2-\theta_1|}-1). \end{aligned} \tag{5.3.2}$$

Denoting $f^*(\pi(\theta))\, e^{-d(\theta)}$ by $\varphi(\theta)$ it follows from (5.3.2) that

$$|\nabla\varphi(\theta)/\varphi(\theta)| \leqq B+1. \tag{5.3.3}$$

Hence, from Lemma 3.4.1 $\varphi^*(x) = \int p_\theta(x)\varphi(\theta)\, d\theta$ satisfies

$$|\nabla\varphi^*(x)/\varphi^*(x)| \leqq B+1 \tag{5.3.4}$$

and

$$\varphi^*(x) \geqq 2^{-m}e^{-B-2}\varphi(x) = 2^{-m}e^{-B-2-d(x)}f^*(\pi(x)). \tag{5.3.5}$$

From Corollary 3.2.3 there is a constant ζ_1 such that

$$|\nabla f^*(x)/f^*(x)| \leqq \zeta_1(1+d(x)). \tag{5.3.6}$$

Substituting the definition of φ, and then using (5.3.5) we see that

$$\begin{aligned} f^*(x) &\leqq e^{-d^2(x)}f^*(\pi(x)) = \exp(-d^2(x)+d(x))\varphi(x) \\ &\leqq 2^m \exp(-d^2(x)+d(x)+B+2)\varphi^*(x) \end{aligned} \tag{5.3.7}$$

$f_1^* = f^*+\varphi^*$. Hence

$$\begin{aligned} |\nabla f_1^*(x)| &= |\nabla f^*(x)+\nabla\varphi^*(x)| \\ &\leqq \zeta_1(1+d(x))f^*(x)+(B+1)\varphi^*(x) \\ &\leqq \zeta_1 f_1^*(x)+\zeta_1 d(x)f^*(x)+\varphi^*(x) \\ &\leqq \zeta_1 f_1^*(x)+k_1\varphi^*(x)+\varphi^*(x) \\ &\leqq B_1 f_1^*(x) \end{aligned} \tag{5.3.8}$$

where here $k_1 = \sup \zeta_1 d(x)\cdot 2^m \exp(-d^2(x)+d(x)+B+2) < \infty$ and $B_1 = \zeta_1+k_1+1$.

It remains only to prove that $\{Z_t^1\}$ is recurrent. We shall do this by showing that for each $\varepsilon_1 > 0$ there exists a piecewise differentiable function, j_1, satisfying

(5.3.9a) $$j_1(x) \geqq 1 \qquad |x| \leqq 1$$

(5.3.9b) $$\lim_{r\to\infty} \sup_{|x|=r} j_1(x) = 0$$

(5.3.9c) $$\int |\nabla j_1(x)|^2 f_1{}^*(x)\,dx < \varepsilon_1.$$

When this is established the recurrence of $\{Z_t^1\}$ then follows immediately from Theorem 4.3.1.

Since $\{Z_t\}$ is recurrent, given $\varepsilon > 0$ there is an $R > 0$ and a piecewise differentiable function j satisfying

$$j(x) \geqq 1 \qquad |x| \leqq 1$$

$$j(x) = 0 \qquad |x| \geqq R$$

$$\int |\nabla j(x)|^2 f^*(x)\,dx < \varepsilon.$$

We remind the reader that $K^1 = \{x : d(x) \leqq 1\}$. Let $\pi_1(x)$ be the projection of x on K^1 and let $d_1(x)$ be the distance of x from K^1. Thus $d_1(x) = \sup(0, d(x)-1)$ and $|x-\pi_1(x)| = d_1(x)$. Given $\varepsilon > 0$, R, and j as above we may transform the variables in the following integral from x to $\pi_1(x)$, $d_1(x)$ to get

(5.3.10) $$\begin{aligned}\varepsilon &\geqq \int_{1<d(x)<2} |\nabla j(x)|^2 f^*(x)\,dx \\ &= \int_{0<d_1<1}\int |\nabla j(x^{-1}(\pi_1, d_1))|^2 f^*(x^{-1}(\pi_1, d_1)) J_1(\pi_1, d_1)\,d\pi_1\,dd_1\end{aligned}$$

where J_1 is the Jacobian of the indicated transformation and $d\pi_1$ is the appropriate Lebesgue differential on the boundary of K^1. The geometry of the situation (especially, K convex) guarantees $1 \leqq J_1 \leqq d^{m-1} = (d_1+1)^{m-1}$. It follows from (5.3.10) that there is a value β, $1 \leqq \beta \leqq 2$, such that

(5.3.11) $$\int |\nabla j(x^{-1}(\pi_1, \beta))|^2 f^*(x^{-1}(\pi_1, \beta))\,d\pi_1 \leqq \varepsilon.$$

With β as above, we consider the set K^β and let $\pi_\beta(x)$ be the projection of x on K^β and $d_\beta(x)$ be the distance of x from K^β. Define (for this section only)

(5.3.12) $$j_1(x) = j(\pi_\beta(x)) \exp(-\alpha d_\beta(x))$$

where $\alpha > 0$ is a small positive number whose value will be specified later. Clearly j_1 satisfies (5.3.9a) and (5.3.9b). We first write

(5.3.13) $$\int |\nabla j_1(x)|^2 f_1{}^*(x)\,dx = \int_{K^\beta} |\nabla j_1(x)|^2 f_1{}^*(x)\,dx + \int_{E^m-K^\beta} |\nabla j_1(x)|^2 f_1{}^*(x)\,dx.$$

It follows from (5.3.4) and Lemma 3.4.1 that $\varphi^*(x) \leqq (\exp(B+1)^2+2^{m/2})\varphi(x)$. From Lemma 3.1.3, $f^*(x) \geqq e^{-(2B+2)} f^*(\pi(x))$ for all $x \in K^\beta$. Hence for $x \in K^\beta$

$$\varphi^*(x) \leqq (\exp(B+1)^2+2^{m/2})\, e^{2B+2} f^*(x).$$

It follows there is a $k_2 < \infty$ such that for all $x \in K^\beta$

(5.3.14) $$f_1^*(x) \leqq k_2 f^*(x).$$

Hence

(5.3.15) $$\int_{K^\beta} |\nabla j_1(x)|^2 f_1^*(x)\,dx \leqq k_2 \int_{K^\beta} |\nabla j(x)|^2 f^*(x)\,dx \leqq k_2\varepsilon.$$

For the second integral on the right of (5.3.13) we begin with the change of variables

(5.3.16) $$\int_{E^m-K^\beta} |\nabla j_1(x)|^2 f_1^*(x)\,dx$$
$$= \int_{d_\beta>0}\int |\nabla j_1(x^{-1}(\pi_\beta, d_\beta))|^2 f_1^*(x^{-1}(\pi_\beta, d_\beta)) J_2(\pi_\beta, d_\beta)\,d\pi_\beta\,dd_\beta$$

where (as in 5.3.10) J_2 is the Jacobian of the indicated transformation and $d\pi_\beta$ is the Lebesgue differential on the boundary of K^β. $J_2 \leqq (d/\beta)^{m-1} \leqq (d_\beta+1)^{m-1}$. From the definition (5.3.12) and a computation similar to that in the first part of (5.3.2)

(5.3.17) $$|\nabla j_1(x)| \leqq |\nabla j(\pi_\beta(x))| \exp(-\alpha d_\beta(x)) + \alpha j(\pi_\beta(x)) \exp(-\alpha d_\beta(x)).$$

We recall $f_1^*(x) = f^*(x) + \varphi^*(x)$. Furthermore, $f^*(x) \leqq e^{-d_\beta{}^2/2} f^*(\pi_\beta(x))$. Combining (5.3.3) and Lemma 3.4.1 we see that

$$\begin{aligned}\varphi^*(x) &\leqq (\exp(B+1)^2 + 2^{m/2})\varphi(x) = (\exp(B+1)^2 + 2^{m/2})\, e^{-d(x)} f^*(\pi(x))\\ &= (\exp(B+1)^2 + 2^{m/2})\, e^{-d(x)}\, \varphi(\pi(x))\\ &\leqq (2\pi)^{m/2}\, e^{B+\frac{3}{2}} (\exp(B+1)^2 + 2^{m/2})\, e^{-d(x)}\, \varphi^*(\pi(x)).\end{aligned}$$

Hence there is a constant k_3' such that

$$f_1^*(x) \leqq k_3'\, e^{-d(x)} f^*(\pi(x)).$$

We note using (5.3.6) that $f^*(\pi_\beta(x)) \geqq e^{-4\zeta_1} f^*(\pi(x))$. Hence there is a constant k_3 such that

(5.3.18) $$f_1^*(x) \leqq k_3\, e^{-d(x)} f^*(\pi_\beta(x)).$$

(5.3.19) $$\begin{aligned}&\int_{E^m-K^\beta} |\nabla j_1(x)|^2 f_1^*(x)\,dx\\ &\quad\leqq k_3 \iint_{d_\beta>0} |\nabla j(\pi_\beta)|^2 \exp(-(1+\alpha)d_\beta) f^*(\pi_\beta)(d_\beta+1)^{m-1} d\pi_\beta dd_\beta\\ &\qquad + k_3\alpha \iint_{d_\beta>0} j(\pi_\beta) \exp(-(1+\alpha)d_\beta)(d_\beta+1)^{m-1} f^*(\pi_\beta) d\pi_\beta\, dd_\beta.\end{aligned}$$

By (5.3.10) the first integration (with respect to π_β) in the first expression on the right of (5.3.19) is $\leqq \varepsilon$. Since $j(x) = 0$ for $|x| \geqq R$ the first integration in the second expression is finite. For $\alpha \leqq 1$

$$\int (d_\beta+1)^{m-1} \exp(-(1+\alpha)d_\beta) dd_\beta \leqq (m-1)!\, e^2.$$

Hence for α sufficiently small

(5.3.20) $$\int_{E^m-K^\beta} |\nabla j_1(x)|^2 f_1^*(x)\,dx \leqq k_3(m-1)!\, e^2\varepsilon + \varepsilon.$$

It follows from (5.3.13), (5.3.15) and (5.3.20) that given ε_1 it is possible to choose ε and α so that with j_1 constructed as above, j_1 has the desired properties (5.3.9). As remarked at (5.3.9) this completes the proof of the lemma.

5.4. *Construction of a smooth minimizing sequence.* We assume throughout this section that (5.1.1) is satisfied. If $K \neq E^m$ let F_1 be as in Lemma 5.3.1 of the preceding section, and if $K = E^m$ define $F_1 = F$. In either case there is a $B_1 < \infty$ such that $|\nabla f_1^*(x)|/f_1^*(x) < B_1$ for all $x \in E^m$, and the diffusion $\{Z_t^1\}$ corresponding to F_1 is recurrent. The result we need is described in the following lemma.

LEMMA 5.4.1. *Let F_1 be as above. Then there exist constants $B_2 < \infty$, $B_3 < \infty$ for which the following is true: For each $\varepsilon > 0$ there is a continuous positive function j, continuously differentiable on $\{x:|x| > 1\}$, such that $j(x) = 1$ for $|x| \leqq 1, j \leqq 1$, and*

(5.4.1) $$j(y)/j(x) \leqq B_2 \exp(B_2|y-x|) \qquad \text{for all } x, y \in E^m$$

and

(5.4.2) $$\int j^2(\theta)F(d\theta) \leqq \int j^2(\theta)F_1(d\theta) < \infty,$$

and

(5.4.3) $$\int|\nabla j(x)|^2 f_1^*(x)\, dx < \varepsilon,$$

and there exists a family $\{q_\theta:\theta \in E^m\}$ of probability densities on E^m such that $q_\theta(x) = 0$ for $|\theta-x| > 1$; $q_\theta(x) \leqq B_3 < \infty$ for all θ, x; and

(5.4.4) $$j(\theta) = \int j(y)q_\theta(y)\, dy.$$

PROOF. For this proof define the measure H by $H(d\theta) = \exp(2B_1|\theta|)\, d\theta$, and define the measures H_i, $i \geqq 1$, by $H_i = F_1 + (1/i)H$. (H_i is not related to the functions $\hat{h}_i$ of Sub-section 1.3.) Using Lemma 3.4.1 we see that

(5.4.5) $$h_i^*(x) > h^*(x)/i \geqq (2\pi)^{-m/2} e^{-2B_1-\frac{1}{2}} e^{2B_1|x|}/i.$$

It then follows from Corollary 4.3.3 that the diffusion generated by H_i is transient.
Again using Lemma 3.4.1, $|\nabla h^*(x)|/h^*(x) \leqq 2B_1$ hence

(5.4.6) $$|\nabla h_i^*(x)|/h_i^*(x) \leqq 2B_1.$$

There is no loss of generality in assuming B_1 has been chosen so that $B_1 > 1$. For convenience we will do so in this proof.

Observe that since $|\nabla f_1^*(x)|/f_1^*(x) < B_1$, $\log f_1^*(x) = O(B_1|x|)$ as $|x| \to \infty$. From 5.4.5 it then follows that $f_1^*(x)/h^*(x) \to 0$ as $|x| \to \infty$. Let $(r, \varphi)(r = |x|)$ denote spherical coordinates in E^m. It can be checked directly from $\nabla h^*(x) = \int (\theta-x) \exp(2B_1|\theta|)p_\theta(x)\, d\theta$ that $(\partial/\partial r)h^*(x)/h^*(x) \to 2B_1$ uniformly in $|x|$ as $|x| \to \infty$. Since $f_1^*(x)/h^*(x) \to 0$ as $|x| \to \infty$ and $|\nabla f_1^*(x)|/f_1^*(x) < B_1$ this implies

(5.4.7) $$\frac{\partial}{\partial r} h_i^*(x)/h_i^*(x) \to 2B_1$$

uniformly in $|x|$ as $|x| \to \infty$. This and (5.4.6) imply $(\partial/\partial\psi)h_i^*(x)/h_i^*(x) \to 0$ uniformly in $|x|$ and ψ as $|x| \to \infty$ for all unit vectors ψ perpendicular to $r = x/|x|$.

Fix i. Let $b_\circ > 1$ be sufficiently large so that for $|x| \geqq b_\circ$

(5.4.8) $$\left|\frac{\partial}{\partial\psi} h_i^*(x)\right| / h_i^*(x) < B_1/4$$

for all unit vectors ψ perpendicular to r. Let $\{Z_t'^{(i)}\}$ denote the diffusion generated by H_i in the prescribed manner, and let

$$\mathscr{K}_i(x) = \Pr\{\inf_{t>0} |Z_t'^{(i)}| \leqq 1 | Z_0'^{(i)} = x\}.$$

(This $\mathscr{K}_i$ is to $\{Z_t'^{(i)}\}$ as $\mathscr{K}$ of (4.1.4) is to $\{Z_t\}$.) $\mathscr{K}_i$ satisfies the equation $\nabla\cdot(h_i^*\nabla\mathscr{K}_i) = 0$ on $\{x: |x| > 1\}$ subject to the boundary condition $\mathscr{K}_i(x) = 1$ for $|x| \leqq 1$. For all x, $\mathscr{K}_i(x) \leqq 1$. Consider the function

(5.4.9) $$\begin{aligned} K_i(x) &= 1 & |x| &\leqq b \\ &= k_1 \textstyle\int_{|x|}^{\infty} (e^{-3B_1 y/2}/y^{m-1})\, dy & |x| &> b \end{aligned}$$

where $k_1^{-1} = \int_{b_\circ}^{\infty} (e^{-3B_1 y|2}/y^{m-1})\, dy$. By rewriting the operator $\nabla\cdot(h_i^*\nabla)$ in spherical coordinates and using (5.4.8) and (5.4.9) it can be checked that

(5.4.10) $$\nabla\cdot(h_i^*\nabla K_i(x)) > 0$$

for $|x| > b_\circ$. (5.4.10) uses $B_1 \geqq \frac{2}{3}$.) Since $K_i(x) \geqq \mathscr{K}_i(x)$ for $|x| = b_\circ$ it follows from (5.4.10) that

(5.4.11) $$\mathscr{K}_i(x) \leqq K_i(x) \qquad \text{for } |x| \geqq b_\circ.$$

[This result is fairly standard, however a quick proof on the lines of our proofs in Section 4 is as follows: (5.4.10) implies $K_i(Z_t'^{(i)})$ is a super-martingale when $|Z_t'^{(i)}| > b_\circ$. $\mathscr{K}_i(Z_t'^{(i)})$ is a martingale on the same region. Start the process at the point x, $|x| > b_\circ$ and let T be the first time $|Z_t'^{(i)}| = b_\circ$. If $T = \infty$ define $J_i(Z_T'^{(i)}) = 0$ and $\mathscr{K}_i(Z_T'^{(i)}) = 0$. Then $K_i(x) \geqq E(K_i(Z_T'^{(i)})) \geqq E(\mathscr{K}_i(Z_T'^{(i)})) = \mathscr{K}_i(x)$.] [Incidentally, (5.4.11) provides an alternate proof that $\{Z_T'^{(i)}\}$ is transient.] Finally, note that since $f_1^*(x) = O(\exp(B_1|x|))$, it follows that $\int_{|\theta-x|<1} F(d\theta) = O(\exp(B_1|x|))$. Hence

(5.4.12) $$\begin{aligned} \int\mathscr{K}_i^2(\theta)F(d\theta) &\leqq \int K_i^2(\theta)F(d\theta) \\ &\leqq \int_{|\theta|<b_\circ+1} K_i^2(\theta)F(d\theta) + \int(\sup_{|\theta-x|<1} K_i^2(\theta) \\ &\quad\cdot \int_{|x-\theta|<1} F(d\theta))\, dx < \infty. \end{aligned}$$

An important additional fact to notice is that (5.4.11) guarantees that $\mathscr{K}_i \in J$ where J is the class of functions described at the beginning of Sub-section 4.1. Hence

(5.4.13) $$\inf_{j\in J} \int |\nabla j(x)|^2 h_i^*(x)\, dx = \int |\nabla\mathscr{K}_i(x)|^2 h_i^*(x)\, dx.$$

(See the remark following Theorem 4.3.1.) Since $\{Z_t^1\}$ is recurrent, given $\varepsilon > 0$ we may find a piecewise differentiable function s and an $R < \infty$ such that

$$s(x) = 1 \qquad |x| \leqq 1$$

$$s(x) = 0 \qquad |x| > R$$

$$\int |\nabla s(x)|^2 f_1^*(x)\,dx < \varepsilon/2.$$

(See Theorem 4.3.1.) For i sufficiently large, say $i = i_\varepsilon$, $h_i^*(x) \leqq 2f_1^*(x)$ for all $x: |x| < R$. Utilizing (5.4.13) we then have

$$\int |\nabla \mathscr{K}_{i_\varepsilon}(x)|^2 h_{i_\varepsilon}^*(x)\,dx \leqq \int |\nabla s(x)|^2 h_{i_\varepsilon}^*(x)\,dx \leqq 2\int |\nabla s(x)|^2 f_1^*(x)\,dx < \varepsilon. \tag{5.4.14}$$

Given $1 > \varepsilon > 0$ set $j(x) = \mathscr{K}_{i_\varepsilon}(x)$. (5.4.14) verifies (5.4.3) and (5.4.12) verifies (5.4.2). The appropriate form of Harnack's inequality (see Serrin (1956)) verifies the existence of a B_2' such that $f^*(y)/f^*(x) \leqq e^{B_2'|y-x|}$ for $|x|, |y| \geqq \frac{3}{2}$ (say). For $0 < \varepsilon \leqq 1$ there is a bound b_2 such that $\inf_{0<\varepsilon\leqq 1} \inf_{|x|<\frac{3}{2}} \mathscr{K}_{i_\varepsilon}(x) > b_2$. Choosing $B_2 = \sup(B_2', b_2^{-1})$ (5.4.1) is satisfied. (B_2 depends only on B_1 and m.) (5.4.4) follows immediately from Lemma 4.4.2. The proof of the lemma is complete.

5.5. *Preparatory lemmas.* We prove here three technical lemmas to prepare for the proof in Sub-section 5.7. The last (and most interesting) of these lemmas is an integration by parts inequality in E^m. (These lemmas could also have been proved in Sub-section 3.2, but since they were not needed before now, we have deferred their proof to this point.)

LEMMA 5.5.1. *Suppose F satisfies* (5.1.1). *For any constant k there is a $\lambda < \infty$ such that for all $x \in E^m$*

$$\int e^{-k|x-\theta|} p_\theta(x) F(d\theta)/f^*(x) \geqq e^{-\lambda d(x)}/\lambda. \tag{5.5.1}$$

PROOF. We use Lemma 3.2.2 to write

$$\int e^{k|\theta-x|} p_\theta(x) F(d\theta)/f^*(x) < \zeta\, e^{\,d(x)}$$

where $1 < \zeta < \infty$ is a constant depending on k. $p_\theta(x)F(d\theta)/f^*(x)$ is the mass element of a probability distribution. It follows using a Chebyshev type argument that for $s = (\log 2\zeta + \zeta d(x))/k$ (so that $e^{ks} = 2\zeta\, e_\xi^{d(x)}$) we have

$$\int_{|\theta-x|\leqq s} p_\theta(x) F(d\theta)/f^*(x) \geqq \tfrac{1}{2}.$$

Hence

$$\int e^{-k|x-\theta|} p_\theta(x) F(d\theta)/f^*(x) \geqq e^{-ks}/2 = (2\zeta)^{-k} e^{-\xi d(x)}/2.$$

The choice $\lambda = 2\cdot(2\zeta)^k$ is certainly sufficient to yield the desired result.

LEMMA 5.5.2. *Given $k_1 < \infty$ there is a $k_2 < \infty$ such that for all $x, \theta \in E^m$*

$$e^{k_1|x-\theta|} p_\theta(x) \leqq k_2 \int_{|\psi|<k_1+1} p_{\theta+\psi}(x)\,d\psi. \tag{5.5.2}$$

PROOF. Without loss of generality we may and shall assume for this proof that $\theta = 0$ and $x = (x_1, 0, \cdots, 0)$, $x_1 \geqq 0$. For this proof only let

$$T = \{\psi : \psi \in E^m, |\psi| < k_1+1, x_1^2 - k_1 x_1 \geqq |x-\psi|^2 - 2k_1^2\}.$$

It is easily checked that $\int_T d\psi > 0$ and $e^{k_1|x-\theta|} p_\theta(x) \leqq e^{2k_1^2} p_{\theta+\psi}(x)$ for all $\psi \in T$. Let $k_2^{-1} = e^{-2k_1^2} \int_T d\psi$. Then

$$k_2 \int_{|\psi|<k_1+1} p_{\theta+\psi}(x)\, d\psi > k_2 \int_T p_{\theta+\psi}(x)\, d\psi$$
$$> k_2\, e^{k_1|x-\theta|} p_\theta(x)\, e^{-2k_1^2} \int_T d\psi = e^{k_1|x-\theta|} p_\theta(x).$$

This completes the proof of the lemma.

LEMMA 5.5.3. *Given $c_1 < \infty$ there is a $c_2 < \infty$ with the following property: Let $j: E^m \to E^1$ be continuously differentiable. Then if $|Z-y| < c_1$*

$$\int (j(y)-j(x))^2 p_Z(x)\, dx \tag{5.5.3}$$
$$\leqq c_2 \iint_{|\xi| \leqq 2c_1+2} |x-\theta|^{1-m} |\nabla j(x)|^2 p_{Z+\xi}(x)\, d\xi\, dx.$$

PROOF. Fix $y \in E^m$. Let $r = |x-y|$ and let φ denote the usual orthogonal angular coordinates in E^m around the point y. (φ is an $(m-1)$ vector.) In short $(r, \varphi) = (r(x), \varphi(x))$ are spherical coordinates around the point y. For convenience normalize φ so that $\int_{|x|<1} dx = \int r^{m-1}\, dr\, d\varphi$. For convenience, let j_S denote j expressed in terms of these coordinates; i.e. $j(x) = j_S(r(x), \varphi(x))$. In the following integrands the symbols r, s, t are real variables $(r, s > 0)$ and $\xi \in E^m$.

$$(j(y)-j(x))^2 = \left(\int_0^{r(x)} |\nabla j_S(s, \varphi(x))|\, ds\right)^2 \tag{5.5.4}$$
$$\leqq r(x) \int_0^{r(x)} |\nabla j_S(s, \varphi(x))|^2\, ds.$$

For any $Z \in E^m$

$$p_Z(x) \leqq (2\pi)^{-m/2} \exp(-(r(x)-r(Z))^2/2).$$

Hence, letting $r(Z) = r_Z < c_1$

$$\int (j(y)-j(x))^2 p_Z(x)\, dx \tag{5.5.5}$$
$$\leqq (2\pi)^{-m/2} \iint r\left(\int_0^r |\nabla j_S(s, \varphi)|^2\, ds\right) \exp(-(r-r_Z)^2/2) r^{m-1}\, dr\, d\varphi$$
$$= (2\pi)^{-m/2} \iint |\nabla j_S(s, \varphi)|^2 \left\{\int_s^\infty r^m \exp(-(r-r_Z)^2/2)\, dr\right\} ds\, d\varphi.$$

Since $r_Z < c_1$ we can choose a c_2' (depending only on m and c_1) such that for all $s > 0$

$$\int_s^\infty r^m \exp(-(r-r_Z)^2/2)\, dr \leqq c_2'(1+s^{m-1}) \exp(-(s-r_Z)^2/2)$$

(see, e.g., Cramér (1946) page 374)). Apply Lemma 5.5.2 to the above noticing that $1+s^{m-1} \leqq 2(m-1)!\, e^s \leqq 2(m-1)! \exp(c_1) \exp(|s-r_Z|)$. It follows that there is a $c_2'' < \infty$ and $c_2 < \infty$ such that for any (s, φ)

$$\int_s^\infty r^m \exp(-(r-r_Z)^2/2)\, dr \leqq c_2'' \int_{|t|<c_1+1} \exp(-(s-t)^2/2)\, dt$$
$$\leqq c_2 (2\pi)^{m/2} \int_{|\xi|<c_1+2} p_{y+\xi}(x^{-1}(s, \varphi))\, d\xi \tag{5.5.6}$$
$$\leqq c_2 (2\pi)^{m/2} \int_{|\xi|<2c_1+2} p_{Z+\xi}(x^{-1}(s, \varphi))\, d\xi.$$

(The next to last step of (5.5.6) requires a justification somewhat similar to that in the proof of Lemma 5.5.2, the details of which we omit. The last step follows directly from the fact that $\{y+\xi:|\xi| < c_1+2\} \subset \{Z+\xi:|\xi| < 2c_1+2\}$.) Combining (5.5.5) and (5.5.6) yields

$$(5.5.7)\quad \int (j(y)-j(x))^2 p_Z(x)\,dx \\ \leqq c_2 \iint s^{1-m} |\nabla j_S(s,\varphi)|^2 \left(\int_{|\xi|<2c_1+2} p_{Z+\xi}(x^{-1}(x,\varphi))\,d\xi\right) s^{m-1}\,ds\,d\varphi.$$

Recalling that $s^{m-1}\,ds\,d\varphi = dx$ and interchanging the order of integration in (5.5.7) yields (5.5.3), which completes the proof of the lemma.

5.6. *On Stein's sufficient condition for admissibility.* In this section we prove a simple extension of Stein's sufficient condition for admissibility (Stein (1955)). See also our discussion in Sub-section 1.3. The basic idea for our version is elementary and was used in R. Farrell (1964), for a similar purpose.

We state the result only for the problem at hand. However the statement and proof clearly generalize to any statistical estimation problem for which the loss function is strictly convex and all the unknown distributions have the same support.

THEOREM 5.6.1. *If there is a sequence of finite nonnegative measures* $\{G_i\}$ *such that* $G_i(\{\theta:|\theta| \leqq 1\}) \geqq 1$ *and*

$$(5.6.1)\qquad B(G_i,\delta)-B(G_i,\delta_{G_i}) \to 0$$

then δ *is admissible.*

PROOF. Suppose δ is not admissible. Then there is a δ' such that $R(\theta,\delta') \leqq R(\theta,\delta)$ and

$$(5.6.2)\qquad \int |\delta'(x)-\delta(x)|\,dx > 0.$$

Define δ'' by $\delta''(x) = (\delta'(x)+\delta(x))/2$. Then, using Jensen's inequality and (5.6.2)

$$\begin{aligned} R(\theta,\delta'') &= \int \|\theta-\delta''(x)\|^2 p_\theta(x)\,dx \\ &< (\tfrac{1}{2}\int \|\theta-\delta(x)\|^2 p_\theta(x)\,dx + (\tfrac{1}{2})\int \|\theta-\delta'(x)\|^2 p_\theta(x)\,dx \\ &= (R(\theta,\delta)+R(\theta,\delta'))/2 \leqq R(\theta,\delta). \end{aligned}$$

$R(\theta,\delta'')$ and $R(\theta,\delta)$ are both continuous functions. Hence (5.6.3) yields the existence of an $\varepsilon > 0$ such that $R(\theta,\delta'') < R(\theta,\delta)-\varepsilon$ for $|\theta| \leqq 1$. Hence if G satisfies $G_i(\{\theta:|\theta| \leqq 1\}) \geqq 1$ we have

$$B(G_i,\delta)-B(G_i,\delta_{G_i}) \geqq B(G_i,\delta)-B(G_i,\delta'') \geqq \varepsilon.$$

This contradicts (5.6.1). It follows that if (5.6.1) is satisfied, δ is admissible.

5.7. *Proof of Theorem* 5.1.1B. We assume without loss of generality that the co-ordinate system has been chosen so that $0 \in \operatorname{supp} F$, and that F has been normalized so that $F(\{\theta:|\theta| < 1\}) \geqq 1$.

Throughout this section we assume the hypotheses of Theorem 5.1.1B are satisfied. We now construct a sequence of finite nonnegative measures G_i, $i = 1, 2, \cdots$ having the following two properties

$$G_i(\{\theta : |\theta| \leqq 1)\} \geqq 1 \tag{5.7.1}$$

and

$$\Delta_i = B(G_i, \delta_F) - B(G_i, \delta_{G_i}) \to 0 \tag{5.7.2}$$

where Δ_i is defined by the above. According to Theorem 5.6.1 the existence of such a sequence will establish the admissibility of δ_F.

As in Sub-section 5.4 if $K = E_m$ let $F_1 = F$ and if $K \neq E^m$ let F_1 be as in Lemma 5.3.1. For this section only, let j_i be the function satisfying the conclusions (5.4.1)–(5.4.3) of Lemma 5.4.1 with $\varepsilon = 1/i$, and let G_i be the measure defined by

$$G_i(d\theta) = j_i^2(\theta) F(d\theta). \tag{5.7.3}$$

(5.4.2) guarantees that G_i is a finite measure. Since $j_i(\theta) \geqq 1$ for $|\theta| \leqq 1$, (5.7.1) is satisfied. [Note that j_i is defined using F_1, but F—not F_1—appears in the expression (5.7.3).] As in Sub-section 1.3 we write $\hat{h}_i(x) = g_i^*(x)/f^*(x)$. Differentiating under the integral sign in the expressions defining g_i^* and f^* and using the fundamental relation (1.2.2) we have

$$\nabla \hat{h}_i(x) = (\int j_i^2(\theta)(\theta - x - \gamma_F(x)) p_\theta(x) F(d\theta))/f^*(x). \tag{5.7.4}$$

Since $\int (\theta - x - \gamma_F(x)) p_\theta(x) F(d\theta) = 0$ we may write

$$\begin{aligned} \|\nabla \hat{h}_i(x)\|^2 &= \|\int (j_1^2(\theta) - j_i^2(x))(\theta - x - \gamma_F(x)) p_\theta(x) F(d\theta)/f^*(x)\|^2 \\ &\leqq (md_1)(\int |j_i^2(\theta) - j_i^2(x)|\, |\theta - x - \gamma_F(x)| p_\theta(x) F(d\theta)/f^*(x))^2. \end{aligned} \tag{5.7.5}$$

Using Cauchy–Schwartz and $|j_i^2(\theta) - j_i^2(x)| = |j_i(\theta) - j_i(x)| \cdot (j_i(\theta) + j_i(x))$ we have

$$\begin{aligned} \|\nabla \hat{h}_i(x)\|^2 \leqq (md_1)&(\int (j_i(\theta) + j_i(x))^2 p_\theta(x) F(d\theta)/f^*(x)) \\ &\cdot (\int (j_i(\theta) - j_i(x))^2 |\theta - x - \gamma_F(x)|^2 p_\theta(x) F(d\theta)/f^*(x)). \end{aligned} \tag{5.7.6}$$

We now turn our attention to the first integral on the right of (5.7.6).

$$\begin{aligned} \hat{h}_i(x) &= \int j_i^2(\theta) p_\theta(x) F(d\theta)/f^*(x) \\ &= j_i^2(x) \int \left(\frac{j_i^2(\theta)}{j_i^2(x)}\right) p_\theta(x) F(d\theta)/f^*(x). \end{aligned}$$

By (5.4.1) $(j_i(\theta)/j_i(x))^2 \geqq B_2^{-2} \exp(-2B_2|x - \theta|)$. Hence by Lemma 5.5.1 there is a λ_1, $1 < \lambda_1 < \infty$, such that $\hat{h}_i(x) \geqq j_i^2(x) \exp(-\lambda_1 d(x))/\lambda_1$.
Thus

$$\int (j_i(\theta) + j_i(x))^2 p_\theta(x) F(d\theta)/f^*(x) \hat{h}_i(x) \leqq 2(1 + \lambda_1\, e^{\lambda_1 d(x)})$$

and we may write

$$(5.7.7)\quad \Delta_i = B(G_i, \delta_F) - B(G_i, \delta_{G_i}) = \int \frac{\|\nabla \hat{h}_i(x)\|^2}{\hat{h}_i(x)} f^*(x)\,dx$$

$$\leqq 4md_1\lambda_1 \iint \exp(\lambda_1 d(x))(j_i(\theta)-j_i(x))^2|\theta - x - \gamma_F(x)|^2 p_\theta(x)F(d\theta)\,dx.$$

Lemma 3.2.3 shows that $|\theta - x - \gamma_F(x)| \leqq |\theta - x| + \zeta_1(d(x)+1)$ for some $\zeta_1 < \infty$. For $\theta \in K$, $d(x) \leqq |\theta - x|$. Hence $\theta \in K$

$$|\theta - x - \gamma_F(x)| \leqq (\zeta_1+1)(|\theta - x|+1) \leqq \zeta_2 \exp(\zeta_2|\theta - x|)$$

where $\zeta_2 = \zeta_1 + 1$. Using this we may rewrite (5.7.7) as

$$\Delta_i \leqq k_1 \iint \exp(k_2|\theta - x|)(j_i(\theta)-j_i(x))^2 p_\theta(x)F(d\theta)\,dx$$

where, here, $k_1 = 4md_1\lambda_1\zeta_2$, $k_2 = \lambda_1 + \zeta_2$. By Lemma 5.5.2 there is a constant k_3 such that

$$\exp(k_2|\theta - x|)p_\theta(x) \leqq k_3 \int_{|\psi| < k_2+1} p_{\theta+\psi}(x)$$

for all θ, x. Thus, interchanging orders of integration,

$$(5.7.8)\qquad \Delta_i \leqq k_1k_3 \int_{|\psi| \leqq k_2+1} \iint (j_i(\theta)-j_i(x))^2 p_{\theta+\psi}(x)\,dxF(d\theta)\,d\psi.$$

We now invoke the property (5.4.4) to write

$$(j_i(\theta)-j_i(x))^2 \leqq \int_{|y-\theta|<1} (j_i(y)-j_i(x))^2 q_\theta(y)\,dy$$

where $\{q_\theta\}$ is defined in Lemma 5.4.1, and satisfies $q_\theta(y) \leqq B_3$ for all $y \in E^m$. Hence

$$(5.7.9)\quad \Delta_i \leqq k_1k_3 \int_{|\psi| \leqq k_2+1} \iint_{|y-\theta|<1} \left(\int (j_i(y)-j_i(x))^2 p_{\theta+\psi}(x)\,dx\right) q_\theta(y)\,dyF(d\theta)\,d\psi.$$

Observe that the integrand in (5.7.9) is only positive on the region $|y-(\theta+\psi)| < k_2+2$. Using Lemma 5.5.3 there is a $k_4 < \infty$ such that

$$(5.7.10)\qquad \Delta_i \leqq k_1k_3k_4 \int_{|\psi| \leqq k_2+1} \iint_{|y-\theta|<1} \left(\iint_{|\xi| \leqq 2k_2+6} |z-y|^{1-m} \cdot |\nabla j_i(z)|^2 p_{\theta+\psi+\xi}(z)\,d\xi\,dZ\right) q_\theta(y)\,dyF(d\theta)\,d\psi.$$

Integrating the above first for the variable y, observing that

$$\int_{|y-\theta|<1} |z-y|^{1-m}\,dy \leqq \int_{|y|<1} y^{1-m}\,dy,$$

and letting

$$k_5 = k_1k_3k_4B_3 \int_{|y|<1} |y|^{1-m}\,dy < \infty,\ k_6 = 2k_2+6$$

$$(5.7.11)\qquad \Delta_i \leqq k_5 \int_{|\psi| \leqq k_6} \int_{|\xi| \leqq k_6} \int |\nabla j_i(z)|^2 \left(\int p_{\theta+\psi+\xi}(z)F(d\theta)\right) dz\,d\xi\,d\psi.$$

Since $p_{\theta+\psi+\xi}(z) = p_\theta(z-\psi-\xi)$ we have

$$\int p_{\theta+\psi+\xi}(z)F(d\theta) = f^*(z-\psi-\xi).$$

Finally, for $|\psi| \leqq k_6$, $|\xi| \leqq k_6$

$$f^*(z-\psi-\xi) \leqq f_1^*(z-\psi-\xi) \leqq \exp(B_1|\psi+\xi|) f_1^*(z) \leqq k_7 f_1^*(z)$$

where $k_7 = \exp(2k_6 B_1)$.

Hence (5.7.11) becomes

$$\begin{aligned} \Delta_i &\leqq k_5 k_7 \int_{|\psi| \leqq k_6} \int_{|\xi| \leqq k_6} \left(\int |\nabla j_i(z)|^2 f_1{}^*(z)\, dz \right) d\xi\, d\psi \\ &= k_8 \int |\nabla j_i(z)|^2 f_1{}^*(z)\, dz \\ &\leqq k_8 / i \end{aligned} \tag{5.7.12}$$

where $k_8 = k_5 k_7 \int_{|\psi| \leqq k_6} \int_{|\xi| \leqq k_6} d\xi\, d\psi$ is independent of i. Thus as $i \to \infty$, $\Delta_i \to 0$, which proves that (5.7.2) is satisfied. It follows from Theorem 5.6.1 that δ_F is admissible. This completes the proof of Theorem 5.1.1B.

6. Various statistical applications.

6.1. *General comments.* W. Strawderman (1969) has studied the following problem: Given a function $\delta(x)$ is there an F such that $\delta = \delta_F$? If $m = 1$ or if δ is spherically symmetric he has obtained an answer to this question, as well as a formula for recovering F from a knowledge of δ. Given an estimator, δ, the first step in determining whether it is admissible according to our Theorem 5.1.1 is to decide whether $\delta = \delta_F$. Since we have nothing to add to the answer provided by Strawderman we will not consider this question further. In the remainder of this Section we therefore begin with the assumption that given an estimator δ whose admissibility is at issue, it is known that $\delta = \delta_F$. A complete knowledge of F is often not necessary since if $\delta = \delta_F$ then $\nabla f^*(x)/f^*(x) = \delta(x) - x$ is automatically known; and admissibility criterion resulting from Theorem 5.1.1 often involve only this quantity.

We take this opportunity to point out that in spite of its broad scope Theorem 5.1.1 does not contain all other known admissibility results for the statistical problem in question. Namely, not all proper Bayes prior distributions satisfy the hypothesis (5.1.1) of Theorem 5.1.1. For example, if $F(d\theta) = p_0(\theta)\, d\theta$ then $f^*(x) = (4\pi)^{-m/2} \exp(-|\theta|^2/4)$ which does not satisfy (5.1.1). On the other hand, it is well known that all proper Bayes procedures are admissible. Hence Theorem 5.1.1 says nothing about such Bayes procedures, which are nevertheless known to be admissible.

6.2. *Admissibility results for $m = 1$.* When $m = 1$ the equation $\mathscr{L}_F j = 0$ (1.3.10) is an ordinary differential equation. An explicit solution of the equation is easy. The general statistical result is given by the following theorem.

THEOREM 6.2.1. *Let $m = 1$. Suppose $\delta = \delta_F$. Suppose either*

$$\int_1^\infty (1/f^*(x))\, dx < \infty \tag{6.2.1}$$

or

$$\int_{-\infty}^{-1} (1/f^*(x))\, dx < \infty \tag{6.2.2}$$

then δ_F is inadmissible.

If both (6.2.1) *and* (6.2.2) *are false* (*i.e., both integrals are infinite*) *and* $\gamma_F(x)$ *is uniformly bounded for* $x \in K$ *then* δ_F *is admissible.*

PROOF. This result in terms of the related diffusions $\{Z_t\}$ is well known. See e.g., Itô and McKean ((1965) Chapter 4). Let $c_+^{-1} = \int_1^\infty f^{*-1}(x)\,dx < \infty$. Then for $x > 1$, $\mathscr{K}(x)$ (see (4.1.4) is given by

$$\mathscr{K}(x) = c_+ \textstyle\int_x^\infty f^{*-1}(t)\,dt < 1$$

as can be seen since $\mathscr{L}_F\mathscr{K} = 0$. Hence $\{Z_t\}$ is transient and δ_F is inadmissible. Similarly, if $c_-^{-1} = \int_{-\infty}^{-1} f^*(t)^{-1}\,dt < \infty$. Conversely, if both (6.2.1) and (6.2.2) are false then $\mathscr{K}(x) \equiv 1$ and $\{Z_t\}$ is recurrent. This is, of course, equally well known, but a direct proof based on the criteria of Theorem 4.3.1 is as follows: Define

$$\begin{aligned} j_i(x) &= 1 && |x| \leqq 1 \\ &= C_{+i}\textstyle\int_x^i f^{*-1}(t)\,dt && 1 < x \leqq i \\ &= C_{-i}\textstyle\int_{-i}^x f^{*-1}(t)\,dt && -i \leqq x < -1 \\ &= 0 && \text{otherwise} \end{aligned}$$

where c_{+i} and c_{-i} are chosen to make j_i continuous. A direct computation yields

$$\int |\nabla j_i(x)|^2 f^*(x)\,dx = c_{+i} + c_{-i} \to 0 \qquad \text{as } i \to \infty.$$

Hence $\{Z_t\}$ is recurrent. The statement of Theorem 6.2.1 then follows directly from Theorem 5.1.1.

The following Corollary gives a more specific criterion based upon Theorem 6.2.1.

COROLLARY 6.2.2. *Let* $m = 1$. *Suppose* $\delta = \delta_F$. *If there is an* $L < \infty$ *and* $k > 1$ *such that* $\gamma_F(x) = \delta_F(x) - x > k/x$ *for* $x > L$ *or* $\gamma_F(x) < -k/x$ *for* $x < -L$ *then* δ_F *is inadmissible. Conversely, if* $\gamma_F(x) \leqq 1/x$ *for* $x > L$ *and* $\gamma_F(x) \geqq -1/x$ *for* $x < -L$ *and* γ_F *is bounded on* K_F *then* δ_F *is admissible.*

PROOF. The proof is immediate from Theorem 6.2.1 after the observation that $f^{*\prime}(x)/f^*(x) = \gamma_F(x) > k/x$ for $x > L$ implies $x^k = 0(f^*(x))$ as $x \to \infty$, etc., and conversely $\gamma_F(x) < 1/x$ for $x > L$ implies $f^*(x) = 0(x)$ as $x \to \infty$.

We can compute directly various admissible estimators for the problem. For example, letting $F(d\theta) = |\theta|\,d\theta$ we have

$$f^*(\theta) = (2/\pi)^{\frac{1}{2}} e^{-\theta^2/2} + \theta \textstyle\int_{-\theta}^{\theta} p_0(t)\,dt$$

from which we compute (for $x \neq 0$)

$$\begin{aligned} \delta_F(x) &= x + f^{*\prime}(x)/f^*(x) \\ &= x + 1/x - \frac{(2/\pi)^{\frac{1}{2}} e^{-x^2/2}}{x(1 - 2\int_{|x|}^\infty p_0(t)\,dt + (2/\pi)^{\frac{1}{2}} e^{-x^2/2})} \end{aligned}$$

as an admissible estimator performing well for large values of $|\theta|$ (in fact, in some sense as well as possible for such values). The admissibility of this estimator is not proved in previous results in the literature.

6.3. *The case of spherical symmetry.* If F, and hence f^*, are spherically symmetric functions then the equation $\mathscr{L}_F j = 0 (j(x) = 1$ for $|x| = 1)$ can be explicitly solved by changing it to spherical co-ordinates, and solving by separation of co-ordinates. The following two results are the direct analogs of those in Subsection 6.2, and we omit their proofs. Note the dependence in these results on dimension m. It is particularly evident in the corollary.

We note first that δ_F is spherically symmetric if and only if F and f^* are; see Strawderman (1969). In this case we may consistently define $f_R{}^*(|x|) = f^*(x)$. For convenience let r_x denote the unit vector in the outward radial direction at the point $x \in E^m$. Hence $\gamma(x)\cdot r_x$ is the radial component of γ at $x \in E^m$.

THEOREM 6.3.1. *Suppose F is spherically symmetric. Hence $f^*(x) = f^*(|x|)$. If*

$$\int_1^\infty (r^{m-1} f_R{}^*(r))^{-1}\, dr < \infty$$

then δ_F is inadmissible. If this integral is infinite and γ_F is bounded then δ_F is admissible.

[Note: Here K_F is either compact or E^m. In the former case, not specifically included in the theorem, δ_F is trivially admissible.]

COROLLARY 6.3.2. *Suppose $\delta = \delta_F$ is spherically symmetric. If there is a $k > 0$ and $L < \infty$ such that*

$$\gamma(x)\cdot r_x \geqq (2-m+k)/|x| \qquad \text{for } |x| > L$$

then δ is inadmissible. Conversely if

$$\gamma(x)\cdot r_x \leqq (2-m)/|x| \qquad \text{for } |x| > L$$

and γ is bounded then δ is admissible.

6.4. *General results in m dimensions.* Contained in this section are several tests for admissibility in the general m dimensional case. Since we cannot solve $\mathscr{L}_F j = 0$ explicitly in this case, none of these results is as encyclopedic as Theorems 6.2.1 or 6.3.1. We retain the notation, r_x, of the previous section. The first result which is really an extension of Corollary 6.3.2 is an obvious consequence of Theorem 6.3.1; hence we call it a corollary.

COROLLARY 6.4.1. *Suppose $\delta = \delta_F$. If for some $k > 0, L < \infty$*

$$\gamma(x)\cdot r_x \geqq (2-m+k)/|x| \qquad \text{for } |x| > L$$

then δ is inadmissible. If

$$\gamma(x)\cdot r_x \leqq (2-m)/|x| \qquad \text{for } |x| > L$$

and γ is uniformly bounded on K_F then δ is admissible.

PROOF. If $\gamma(x)\cdot r_x > (2-m+k)/|x|$ for $|x| > L$ then $f^*(x) > k_1|x|^{2-m+k}$ for $|x| > L$ and some $k_1 > 0$. It follows from Theorem 6.3.1 by the nature of Theorem 5.1.1 that δ is inadmissible. Similarly for the second half of the corollary.

For the next two results we use the following definitions: Let μ_r denote the uniform probability measure on the surface of the sphere $\{x: |x| = r\}$, and let

$$\bar{f}(r) = \int f^*(x)\mu_r(dx) \tag{6.4.1}$$

be the average value of f^* on the surface of the sphere of radius r.

THEOREM 6.4.2. *Suppose $\delta = \delta_F$ and suppose $\gamma(x) = 0(1/|x|)$ as $|x| \to \infty$. Then if*

$$\int_1^\infty (r^{m-1}\bar{f}(r))^{-1}\, dr \dot{<} \infty \tag{6.4.2}$$

δ is inadmissible and if

$$\int_1^\infty (r^{m-1}\bar{f}(r))^{-1}\, dr = \infty \tag{6.4.3}$$

then δ is admissible.

REMARK. It will be evident from the proof that $f^*(x_r)$ where x_r is any member of the set $\{x: |x| = r\}$ may be substituted for $\bar{f}(r)$ in the criteria (6.4.2) and (6.4.3). Hence $\bar{f}(r)$ need not actually be computed.

PROOF. For an appropriate $k_1 < \infty$, $|\nabla f^*(x)|/f^*(x) < k_1/|x|$ for all x. Let $x_r, y \in \{x: |x| = r\}$. Then since $|y-x_r| \leqq 2\pi r$ it follows that

$$e^{-2\pi k_1} \leqq f^*(y)/f^*(x_r) \leqq e^{2\pi k_1}.$$

Hence

$$\inf_{\{y:|y|=r\}} f^*(y) \geqq e^{-2\pi k_1} f^*(x_r) \tag{6.4.4}$$

and

$$\sup_{\{y:|y|=r\}} f^*(y) \leqq e^{2\pi k_1} f^*(x_r). \tag{6.4.5}$$

$\bar{f}(r)$ may be substituted for $f^*(x_r)$ in (6.4.4) and (6.4.5). It follows that

$$\int |\nabla j(x)|^2 f^*(x)\, dx \geqq e^{-2\pi k_1} \int (\nabla j(x)\cdot r_x)^2 \bar{f}(|x|)\, dx.$$

By the same arguments as in Theorem 6.3.1 it follows that if (6.4.2) holds then δ is inadmissible. A similar argument will prove the admissibility part of the Theorem which completes the proof.

Half of the above result remains valid without the assumption that $\gamma(x) = 0(1/|x|)$. The following yields an interesting test for recurrence of $\{Z_t\}$ as well as for the admissibility of δ_F.

THEOREM 6.4.3. *Suppose $\delta = \delta_F$. Suppose γ is uniformly bounded on K_F and suppose*

$$\int_1^\infty (r^{m-1}\bar{f}(r))^{-1}\, dr = \infty. \tag{6.4.6}$$

Then δ is admissible.

PROOF. Define

$$\begin{aligned} j_i(x) &= 1 && |x| \leqq 1 \\ &= c_i \int_{|x|}^{i} (r^{m-1}\bar{f}(r))^{-1}\, dr && 1 < |x| \leqq i \\ &= 0 && |x| > i \end{aligned}$$

where $c_i = (\int_1^i (r^{(m-1)}\bar{f}(r))^{-1}\, dr)^{-1} \to 0$ as $i \to \infty$. Then substituting and changing to spherical co-ordinates we have

$$\begin{aligned} \int |\nabla j(x)|^2 f^*(x)\, dx &= \int_1^i c_i^2(|x|^{m-1}\bar{f}(|x|)^{-2} f^*(x)\, dx \\ &= c_i^2 \int_1^i r^{1-m}\bar{f}^{-1}(r)\, dr = c_i \to 0. \end{aligned}$$

Hence δ is admissible and $\{Z_t\}$ is recurrent.

There is also an easy result in the opposite direction in the same spirit as Theorem 6.4.3. Let (r, φ) denote spherical co-ordinates in E^m. Let $f_R^*(r, \varphi) = f^*(x(r, \varphi))$, etc.

THEOREM 6.4.4. *Suppose $\delta = \delta_F$. If there is a $Q \subset \{\varphi\}$ with $\int_Q d\varphi > 0$ such that*

$$\sup_{\varphi \in Q} \int (r^{m-1} f_R^*(r, \varphi))^{-1}\, dr < \infty \tag{6.4.7}$$

then δ_F is inadmissible.

REMARK. (6.4.7) is essentially equivalent to $\int r^{1-m} \int_Q (f_R^*(r, \varphi))^{-1}\, d\varphi\, dr < \infty$. Since

$$\int_Q (f_R^*(r, \varphi))^{-1}\, d\varphi \geqq (\int f_R^*(r, \varphi)\, d\varphi)^{-1} \int_Q d\varphi),$$

it will be seen that it is possible to construct examples where neither (6.4.6) nor (6.4.7) are satisfied.

PROOF. Observe that if $j(x) = 1$ for $|x| \leqq 1$, and $j(x) = 0$ for $|x| > R$,

$$\begin{aligned} \int |\nabla j(x)|^2 f^*(x)\, dx & \\ \geqq \int_Q \int \left|\frac{\partial}{\partial r} j_R(r, \varphi)\right|^2 r^{m-1} f_R^*(r, \varphi)\, dr\, d\varphi & \\ \geqq \int_Q (\int (r^{m-1} f_R^*(r, \varphi))^{-1}\, dr)^{-1}\, d\varphi. & \end{aligned} \tag{6.4.8}$$

(6.4.8) and (6.4.7) imply via Theorems 5.1.1 and 4.3.1 that δ_F is inadmissible.

We note there are other criteria which can be proved by variations of the above arguments—for example, if γ_F is uniformly bounded and there is a φ such that $\int (f_R^*(r, \varphi))^{-1}\, dr < \infty$ then δ_F is inadmissible—but we will not pause here to catalog further results of this type.

6.5. *Co-ordinate by co-ordinate estimation.* Let $\delta_1(x) = x$ denote the usual estimator for $x \in E^1$, i.e., for dimension $m = 1$. Suppose $m > 1$ so that we observe $x = (x_1, x_2, \cdots, x_m)$ where the x_i are observations from independent normal (0, 1) distributions. Consider, for example, usual squared error loss—i.e., $\|t\| = |t| =$

$\sum t_i^2$. Then there is an m—namely any $m \geqq 3$—such that the estimator $\delta_1{}^{(m)}(x) = (\delta_1(x_1), \cdots, \delta_1(x_m))$ is inadmissible. This raises the following question: Is there an estimator δ_2, say, on dimension 1 such that for all m the estimator $\delta_2{}^{(m)}$ defined by $\delta_2{}^{(m)}(x) = (\delta_2(x_1), \delta_2(x_2), \cdots, \delta_2(x_m))$ is admissible? If δ_2 is proper Bayes then, of course, $\delta_2{}^{(m)}$ is also proper Bayes—hence admissible. Thus the interesting question is whether there is a non-proper Bayes estimator δ_2 for which $\delta_2{}^{(m)}$ is admissible for all m. The following theorem answers this question in the affirmative.

THEOREM 6.5.1. *Suppose δ is a generalized Bayes estimator on E^1 such that $\delta = \delta_F$ where $f^*(x) = O(1/|x|)$ and γ is bounded on K_F. Let $\delta^{(m)}$ be defined on E^m by $\delta^{(m)}(x) = (\delta(x_1), \cdots, \delta(x_m))$. Then for any m, $\delta^{(m)}$ is admissible.*

REMARK. That estimators such as δ exist is, of course, verified by Lemma 3.4.1; simply choose $F(d\theta) = f(\theta)\, d\theta$ where $f(\theta) = O(1/|\theta|)$.

PROOF. It is a matter of m-dimensional calculus to check that $\bar{f}(r) = O(1/|r|^{m-\alpha})$ for any $\alpha > 0$ where $\bar{f}$ is defined by (6.4.1). Hence by Theorem 6.4.3, $\delta^{(m)}$ is admissible for all m. The proof is complete.

A more general question is the following: Suppose δ_2 is an admissible estimator for $m-1$ dimensions. Consider an m-dimensional problem. Suppose we wish to use δ_2 to estimate the first $m-1$ components of θ on the basis of the first $m-1$ components of the observation x, and estimate θ_m independently by an estimator based on x_m alone. Can the resulting procedure be admissible? More formally, we ask if there is an admissible estimator of the form $\delta'(x) = (\delta_2(x_1, \cdots, x_{m-1}), \delta_3(x_m))$. It can be shown that if δ_3 is a proper Bayes estimator then δ' is admissible. Hence there are many choices of δ_3 such that for any admissible δ_2 the estimator δ' is admissible. A proof of this fact is contained in Theorem 6.5.2. This proof involves only the appropriate form of Stein's necessary and sufficient condition; not our Theorem 5.1.1.

Two questions which remain are:

(a) Given a δ_2 can one find a δ_3 which is not genuine Bayes such that δ' is admissible? and

(b) Is there a δ_3 which is not genuine Bayes such that for any admissible δ_2 the estimator δ' is admissible?

In view of Theorem 6.5.2 below, one would expect that the answer to the first question is "Yes." In the case where $\delta_2(x)-x = O(1/|x|)$ the results of the preceding section can be used to prove that the answer is in fact, "Yes." We do not give here the proof of this fact. The second question appears to us the more interesting but we have been unable to answer it, even if the condition is added that $\delta_1(x)-x = O(1/|x|)$.

THEOREM 6.5.2. *Let δ_2 be any admissible estimator on E^{m-1} and δ_3 be any proper Bayes estimator on E^1. Define the estimator δ' on E^m by $\delta'((x_1, \cdots, x_m)) = (\delta_2(x_1, \cdots, x_{m-1}), \delta_3(x_m))$. Then δ' is admissible.*

PROOF. Since δ_2 is admissible there is a sequence of finite nonnegative measures $\{G_i\}$ on E^{m-1} such that $G_i(\{0\}) = 1$ and $R(G_i, \delta_2) - R(G_i, \delta_{G_i}) \to 0$. Suppose δ_3 is Bayes for the prior probability distribution H, i.e., $\delta_3 = \delta_H$. Define the sequence H_i of measures on E^m by

$$H_i(A \times B) = G_i(A)H(B), \qquad A \subset E^{m-1}, B \subset E^1.$$

Then, since H is a probability distribution and $\delta_3 = \delta_H$,

$$R(H_i, \delta') - R(H_i, \delta_{H_i}) = R(G_i, \delta_2) - R(G_i, \delta_{G_i}) \to 0.$$

Furthermore there will be a sphere S of radius one about some point such that $\inf_i H_i(S) > 0$. It follows as in Sub-section 5.6 that δ' is admissible.

There is an alternate probabilistic argument leading to the above theorem via our Theorem 5.1.1. Denote by $\{Z_t^{(1)}\}$, $\{Z_t^{(2)}\}$, and $\{Z_t^{(3)}\}$ the diffusions associated with the estimators δ', δ_2 and δ_3, respectively. It is a general fact that if $\{Z_t\}$ is the diffusion associated with the measure F then the (left) invariant measure for the process $\{Z_t\}$ is $f^*(x)\,dx$. Thus if $\delta_3 = \delta_H$ where H is a probability distribution it follows that the invariant measure for $\{Z_t^{(3)}\}$ is $h^*(x)\,dx$, also a probability measure. Hence $\{Z_t^{(3)}\}$ is ergodic. It follows that if $\{Z_t^{(2)}\}$ is any recurrent diffusion on E_{m-1} then the diffusion $\{Z_t^{(1)}\} = \{(Z_t^{(2)}, Z_t^{(3)})\}$ on E^m must also be recurrent. Hence δ' is admissible.

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